

Frame-Like Action and Unfolded Formulation for Massive Higher-Spin Fields

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Abstract

Unfolded equations of motion for symmetric massive bosonic fields of any spin in Minkowski and $(A)dS$ spaces are presented. Manifestly gauge invariant action for a spin $s \geq 2$ massive field in any dimension is constructed in terms of gauge invariant curvatures.

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1 Introduction

Although elementary particles of spins higher than two are unlikely to be directly observed in the modern high-energy experiments, there are several reasons to believe that they play a fundamental role at ultra high energies relevant to quantum gravity. Indeed, infinite towers of massive higher-spin (HS) fields are important for consistency of String Theory [1, 2, 3]. On the other hand, existence of nonlinear theories of massless HS fields with unbroken HS symmetries indicates new remarkable structures underlying a theory of fundamental interactions (for recent reviews on HS theory see, e.g., [4, 5, 6, 7, 8, 9]). To uncover relation between the two types of theories and, more generally, to understand a spontaneous breakdown mechanism for HS symmetries that gives rise to a theory of HS massive fields, it is important to extend the approach, which underlies nonlinear massless HS theories, to massive HS fields starting from the linearized level. This is the goal of this paper.

There are two approaches to HS fields. The metric-like approach [10, 11, 12, 13, 14, 15, 16] generalizes metric formulation of gravity. The frame-like formalism [17, 18, 19, 20, 21, 22], that generalizes Cartan formulation of gravity, deals with differential forms. It reproduces the metric-like formulation in a particular gauge and is most appropriate to control gauge HS symmetries and interactions [23, 24, 25]. The study of the frame-like approach in the context

of massive HS fields was initiated by Zinoviev who constructed gauge invariant actions in terms of the frame-like fields in [22] where, however, the gauge invariance of the action was not manifest, requiring the adjustment of coefficients in the action from the condition of gauge invariance. In this paper we propose the manifestly gauge invariant action for symmetric massive HS fields formulated in terms of gauge invariant HS curvatures. The same time, these curvatures underly the construction of unfolded formulation of free massive HS field equations which is also given in this paper.

Free equations for symmetric massive fields of any spin are¹ [26]

$$(\partial^n \partial_n + m^2) \phi_{a(s)} = 0, \quad \partial^n \phi_{na(s-1)} = 0, \quad \phi^n_{na(s-2)} = 0. \quad (1.1)$$

Our aim is to reformulate the equations (1.1) in the unfolded form of generalized zero-curvature equations [27] (for a review see [7])

$$R^\alpha(x) \stackrel{def}{=} dW^\alpha(x) + G^\alpha(W(x)) = 0, \quad (1.2)$$

where $d = dx^\mu \partial_\mu$ is the exterior differential and

$$G^\alpha(W^\beta) \stackrel{def}{=} \sum_{n=1}^{\infty} f_{\beta_1 \dots \beta_n}^\alpha W^{\beta_1} \dots W^{\beta_n} \quad (1.3)$$

is built from the exterior product (which is implicit in this section) of differential forms $W^\beta(x)$ and satisfies the compatibility condition

$$G^\beta(W) \frac{\delta^L G^\alpha(W)}{\delta W^\beta} \equiv 0. \quad (1.4)$$

Here index α enumerates a set of differential forms, which may be infinite.

Unfolding consists of the equivalent reformulation of a system of partial differential equations in the form (1.2), which is always possible by virtue of introducing enough (may be infinite towers of) auxiliary and Stueckelberg fields. The fields, that neither can be expressed in terms of derivatives of other fields, nor can be gauged away are called *dynamical*. *Auxiliary fields* are expressed via derivatives of all orders of the dynamical field $\phi_{a(s)}$. *Stueckelberg fields* are pure gauge with respect to algebraic shift symmetries. Differential conditions on the dynamical fields imposed by the unfolded equations are called *dynamical equations*. (Note that the standard tool for the analysis of physical content of unfolded equations is provided by the σ_- -cohomology technics [28, 7].)

Unfolded field equations (1.2) are manifestly invariant under the gauge transformation with a degree $p^\alpha - 1$ differential form gauge parameter $\varepsilon^\alpha(x)$ associated to any degree $p^\alpha > 0$ form W^α

$$\delta W^\alpha = d\varepsilon^\alpha - \varepsilon^\beta \frac{\delta^L G^\alpha(W)}{\delta W^\beta} \quad (1.5)$$

because

$$\delta R^\alpha = -R^\gamma \frac{\delta^L}{\delta W^\gamma} \left(\varepsilon^\beta \frac{\delta^L G^\alpha(W)}{\delta W^\beta} \right). \quad (1.6)$$

¹ For notation see Appendix A

The gauge transformations of 0-forms only contain the gauge parameters ε^{α_1} associated to 1-forms W^{α_1} in (1.5):

$$\delta C^{\alpha_0} = -\varepsilon^{\alpha_1} \frac{\delta^L G^{\alpha_0}(W)}{\delta W^{\alpha_1}}. \quad (1.7)$$

In the unfolded dynamics approach, background Minkowski or $(A)dS$ geometry is described by the 1-form frame field (vielbein) $e^a = e^a_\mu dx^\mu$ and Lorentz spin-connection 1-form $\omega^{a,b} = \omega^{a,b}_\mu dx^\mu$, $\omega^{a,b} = -\omega^{b,a}$, that obey the equations

$$T^a \equiv de^a + \omega^{a,b} \wedge e^b = 0, \quad (1.8)$$

$$R^{a,b} \equiv d\omega^{a,b} + \omega^{a,c} \wedge \omega^{c,b} - \lambda^2 e^a \wedge e^b = 0, \quad (1.9)$$

where $-\lambda^2$ is the cosmological constant. The vielbein e^a_μ is nondegenerate and relates fiber (tangent) and base (world) indices. Eq. (1.8) is the zero-torsion condition that expresses the Lorentz connection $\omega^{a,b}$ via vielbein e^a . The equation (1.9) then describes the AdS_d space with the symmetry algebra $o(d-1,2)$ for $\lambda^2 > 0$, dS_d space with the symmetry algebra $o(d,1)$ for $\lambda^2 < 0$ and Minkowski space for $\lambda^2 = 0$.

Unfolded formulation brings together such important issues as coordinate independence due to using the exterior algebra formalism and manifest gauge invariance (Stueckelberg in the massive case) that, controls a number of degrees of freedom in the system. The unfolded dynamics approach made it possible to find a full nonlinear system of equations for massless fields in four [24] and any [25] dimensions. Extension of the unfolded description to the massive case should shed light on the mechanism of spontaneous breakdown of HS gauge symmetries and, eventually, unfolded formulation of String Theory to establish a correspondence between the latter and HS theory.

Our aim is to find an explicit form of the unfolded system of equations that contains a dynamical equation (1.1) on the dynamical field $\phi_{a(s)}$ plus constraints that express auxiliary fields in terms of $\phi_{a(s)}$. The unfolded form of massive field equations was worked out in [28] for the case of spin 0. General features of the unfolded description of massive fields of any spin were analyzed in [29].

In this paper we present the explicit form of the unfolded equations for the very particular case of spin s massive symmetric gauge fields in Minkowski and $(A)dS$ space which, as we show, can be described by a set of 1-form connections and 0-form Weyl tensors valued in various Lorentz tensor representations described by two-row Young tableaux. In contrast to massless and partially massless HS fields and in agreement with the results of [29], gauge invariant curvatures for massive fields necessarily involve 0-forms. Using the gauge invariant HS curvatures that result from the unfolded formulation of massive fields we also present the manifestly gauge invariant actions for HS massive fields.

The layout of the rest of the paper is as follows. Unfolded equations, that describe Weyl module of a spin s massive field, are derived in Section 2. Extension of these equations to 1-form gauge potentials is given in Section 3. Manifestly gauge invariant action for massive HS fields is constructed in Section 4. Section 5 contains conclusions. Our conventions and some useful formulas are given in Appendices A and B, respectively.

2 Forms of definite degree

2.1 General form of curvatures

The frame-like formulation of massless [17, 18, 19, 20] and partially massless [21] fields operates with 1-forms valued in irreducible Lorentz tensor spaces described by two-row Young tableaux (for more details on Young tableaux see, e.g., [7, 30]; the facts relevant to the consideration of this paper are given in Appendix A). Let $\mathbf{Y}(k, l)$ denote the space of tensors described by two-row Young tableaux with k cells in the first row and $l \leq k$ cells in the second row. In this section we analyze the most general linear unfolded equations that can be formulated in terms of p -forms $W^{a(k), b(l)}$ with some definite p , valued in $V = \sum_{k \geq l \geq 0} \oplus \mathbf{Y}(k, l)$. Since the analysis is insensitive to p , it is kept arbitrary in this section.

Unfolded equations formulated for differential forms of a definite degree, valued in V , read as

$$0 = R^{a(k), b(l)} = DW^{a(k), b(l)} + G^{a(k), b(l)}(W, e), \quad (2.1)$$

where D is the Lorentz covariant derivative

$$DC^{ab\dots} = dC^{ab\dots} + \omega^a{}_c C^{cb\dots} + \omega^b{}_c C^{ac\dots} + \dots \quad (2.2)$$

From (1.9) it follows that

$$D^2 C^{a(k), b(l)} = \lambda^2 (k e^a e_c C^{ca(k-1), b(l)} + l e^b e_c C^{a(k), cb(l-1)}). \quad (2.3)$$

The background vielbein 1-form $e^a = dx^\mu e_\mu^a$ enters $G^{a(k), b(l)}(W, e)$ linearly in (2.1). Generally $e^a \wedge W^{\dots}$ is a $(p+1)$ -form valued in a reducible Lorentz module. Since vielbein carries one fiber index, the projection to irreducible components described by two-row Young diagrams gives

$$\begin{array}{|c|} \hline k \\ \hline l \\ \hline \end{array} \otimes \square = \overset{\sigma_+^1}{\begin{array}{|c|} \hline k+1 \\ \hline l \\ \hline \end{array}} \oplus \overset{\sigma_+^2}{\begin{array}{|c|} \hline k \\ \hline l+1 \\ \hline \end{array}} \oplus \overset{\sigma_-^1}{\begin{array}{|c|} \hline k-1 \\ \hline l \\ \hline \end{array}} \oplus \overset{\sigma_-^2}{\begin{array}{|c|} \hline k \\ \hline l-1 \\ \hline \end{array}} + \dots,$$

where tensors described by three-row Young tableaux are projected out. Hence, there exist four linearly independent operators built from the vielbein 1-form. σ_+^1 and σ_+^2 (σ_-^1 and σ_-^2) increase (decrease), respectively, the lengths of the first and the second row by one. Operators σ will be sometimes referred to as “cell operators”.

Thus, the curvature (2.1) can be represented in the form

$$R(W) = DW + \sigma_+^1(W) + \sigma_+^2(W) + \sigma_-^1(W) + \sigma_-^2(W), \quad (2.4)$$

where the form of the cell operators is determined up to overall coefficients by the properties of traceless two-row Young diagrams

$$\sigma_-^1(k, l)(W^{a(k), b(l)}) = f(k, l)(e_m W^{a(k-1)m, b(l)} + \frac{l}{k-l+1} e_m W^{a(k-1)b, b(l-1)m}), \quad (2.5)$$

$$\sigma_-^2(k, l)(W^{a(k), b(l)}) = F(k, l)(e_m W^{a(k), b(l-1)m}), \quad (2.6)$$

$$\sigma_+^1(k, l)(W^{a(k), b(l)}) = g(k, l)(k+1)(e^a W^{a(k), b(l)} - \frac{k}{d+2k-2} e_m W^{a(k-1)m, b(l)} \eta^{aa} -$$

$$- \frac{l}{d+k+l-3} e_m W^{a(k), mb(l-1)} \eta^{ab} + \frac{lk}{(d+2k-2)(d+k+l-3)} e_m W^{a(k-1)b, b(l-1)m} \eta^{aa}, \quad (2.7)$$

$$\begin{aligned} \sigma_+^2(k, l)(W^{a(k), b(l)}) &= G(k, l)(l+1)(e^b W^{a(k), b(l)} - \frac{k}{k-l} e^a W^{a(k-1)b, b(l)} - \\ &- \frac{l}{d+2l-4} e_m W^{a(k), b(l-1)m} \eta^{bb} - \frac{k(k-l-1)}{(k-l)(d+k+l-3)} e_m W^{a(k-1)m, b(l)} \eta^{ab} + \\ &+ \frac{lk(d+2k-4)}{(k-l)(d+2l-4)(d+k+l-3)} e_m W^{a(k-1)b, b(l-1)m} \eta^{ab} + \\ &+ \frac{k(k-1)}{(k-l)(d+k+l-3)} e_m W^{a(k-2)mb, b(l)} \eta^{aa} - \\ &- \frac{lk(k-1)}{(k-l)(d+k+l-3)(d+2l-4)} e_m W^{a(k-2)bb, b(l-1)m} \eta^{aa}). \end{aligned} \quad (2.8)$$

To rule out the terms with Young diagrams $\mathbf{Y}(k, l)$ with $l > k$, that are zero, it is convenient to demand

$$f(k, l) = F(k, l) = g(k, l) = G(k, l) = 0 \quad \text{for } l > k \quad (2.9)$$

and

$$f(n, n) = G(n, n) = 0. \quad (2.10)$$

The compatibility condition (1.4) for the equation (2.4) is

$$D^2 + \sigma^2 = 0, \quad \sigma = \sigma_-^1 + \sigma_-^2 + \sigma_+^1 + \sigma_+^2. \quad (2.11)$$

Restriction of (2.11) to different types of Young tableaux yields the following conditions

$$(\sigma_-^1)^2 = 0, \quad (\sigma_-^2)^2 = 0, \quad (\sigma_+^1)^2 = 0, \quad (\sigma_+^2)^2 = 0, \quad (2.12)$$

$$\{\sigma_-^1, \sigma_-^2\} = 0, \quad \{\sigma_-^1, \sigma_+^2\} = 0, \quad \{\sigma_-^2, \sigma_+^1\} = 0, \quad \{\sigma_+^1, \sigma_+^2\} = 0, \quad (2.13)$$

$$D^2 + \{\sigma_-^1, \sigma_+^1\} + \{\sigma_-^2, \sigma_+^2\} = 0. \quad (2.14)$$

The conditions (2.12) are trivially satisfied due to the antisymmetry of the exterior product. The conditions (2.13) give the following constraints on the coefficients $F(k, l)$, $G(k, l)$, $f(k, l)$ and $g(k, l)$

$$\begin{aligned} \frac{f(k, l-1)F(k, l)}{f(k, l)F(k-1, l)} &= \frac{k-l}{k-l+1} : \quad \{\sigma_-^1, \sigma_-^2\} = 0, \\ \frac{G(k-1, l)f(k, l)}{G(k, l)f(k, l+1)} &= \frac{(k-l-1)(k-l+1)(d+k+l-2)}{(k-l)(k-l)(d+k+l-3)} : \quad \{\sigma_-^1, \sigma_+^2\} = 0, \\ \frac{F(k+1, l)g(k, l)}{F(k, l)g(k, l-1)} &= \frac{d+k+l-3}{d+k+l-2} : \quad \{\sigma_-^2, \sigma_+^1\} = 0, \\ \frac{g(k, l+1)G(k, l)}{g(k, l)G(k+1, l)} &= \frac{k-l+2}{k-l+1} : \quad \{\sigma_+^2, \sigma_+^1\} = 0. \end{aligned} \quad (2.15)$$

Note that only three of these conditions turn out to be independent.

2.2 Field redefinition ambiguity

Eq. (2.4) has the freedom in the field redefinition

$$W \rightarrow \tilde{W}, \quad W^{a(k),b(l)} = \beta(k,l) \tilde{W}^{a(k),b(l)}, \quad \beta(k,l) \neq 0, \quad (2.16)$$

which induces the following redefinition of the coefficients

$$\begin{aligned} \tilde{g}(k,l) &= g(k,l) \frac{\beta(k,l)}{\beta(k+1,l)}, & \tilde{G}(k,l) &= G(k,l) \frac{\beta(k,l)}{\beta(k,l+1)}, \\ \tilde{f}(k,l) &= f(k,l) \frac{\beta(k,l)}{\beta(k-1,l)}, & \tilde{F}(k,l) &= F(k,l) \frac{\beta(k,l)}{\beta(k,l-1)}. \end{aligned} \quad (2.17)$$

For the future convenience of the analysis of variation of the action in Section 4 we fix this ambiguity by demanding the operators $\sigma_+^{1,2}$ be conjugated to $\sigma_-^{1,2}$ up to some factors n and N with respect to the scalar product

$$\langle \psi_{\{p\}}^{a(k),b(l)} | \phi_{\{q\}}^{a(k),b(l)} \rangle = \int \epsilon_{l_1 \dots l_d} e^{l_5} \dots e^{l_d} \psi_{\{p\}}^{l_1 a(k-1), l_2 b(l-1)} \phi_{\{q\}}^{l_3 a(k-1), l_4 b(l-1)}, \quad (2.18)$$

where $\epsilon_{l_1 \dots l_d}$ is the totally antisymmetric tensor. The scalar product $\langle \dots | \dots \rangle$ is nonzero provided that the p -form $\psi_{\{p\}}^{a(k),b(l)}$ and q -form $\phi_{\{q\}}^{a(k),b(l)}$ have $p+q=4$ and carry equivalent representations of the Lorentz group $\mathbf{Y}(k,l)$, $k \geq l \geq 1$ (otherwise the r.h.s. of (2.18) does not make sense).

We impose the following conditions

$$f(k+1,l) = n(k,l) \frac{(d+k-2)k}{k+1} g(k,l), \quad k \geq 1, l \geq 0, \quad (2.19)$$

$$F(k,l+1) = N(k,l) \frac{(d+l-3)(k-l+1)}{(k-l)} G(k,l), \quad k \geq 1, l \geq 0, \quad (2.20)$$

where the coefficients $n(k,l)$ and $N(k,l)$ will be determined later on. For $k \geq 1$ and $l \geq 1$ these conditions follow from

$$\langle \sigma_-^1 \psi_{\{p\}}(k+1,l) | \phi_{\{q\}}(k,l) \rangle = n(k,l) (-1)^p \langle \psi_{\{p\}}(k+1,l) | \sigma_+^1 \phi_{\{q\}}(k,l) \rangle, \quad k \geq 1, l \geq 1,$$

$$\langle \sigma_-^2 \psi_{\{p\}}(k,l+1) | \phi_{\{q\}}(k,l) \rangle = N(k,l) (-1)^p \langle \psi_{\{p\}}(k,l+1) | \sigma_+^2 \phi_{\{q\}}(k,l) \rangle, \quad k \geq 1, l \geq 1, \quad (2.21)$$

while for $k \geq 1, l=0$ they are imposed by hand. Note that eq. (2.20) can be assumed to be satisfied at $k=l=0$ as well because $F(0,1)=G(1,1)=0$ by (2.9), (2.10). However, the condition (2.19) cannot be imposed at $k=l=0$ to relate $f(1,0)$ and $g(0,0)$, because it would imply that $f(1,0)=0$ that is not necessarily true. Instead, it is convenient to impose the condition

$$f(1,0) = n(0,0)g(0,0). \quad (2.22)$$

Let us note that Eqs. (2.19), (2.20) give

$$\frac{g(k,l+1)f(k+1,l)G(k,l)F(k+1,l+1)}{g(k,l)f(k+1,l+1)G(k+1,l)F(k,l+1)} = \frac{N(k+1,l)n(k,l)}{N(k,l)n(k,l+1)} \frac{(k-l)(k-l+2)}{(k-l+1)^2}. \quad (2.23)$$

On the other hand, from (2.15) it follows that

$$\frac{g(k, l+1)f(k+1, l)G(k, l)F(k+1, l+1)}{g(k, l)f(k+1, l+1)G(k+1, l)F(k, l+1)} = \frac{(k-l)(k-l+2)}{(k-l+1)^2}. \quad (2.24)$$

So, the conditions (2.21) require

$$\frac{N(k+1, l)n(k, l)}{N(k, l)n(k, l+1)} = 1. \quad (2.25)$$

Once (2.25) is true, Eq. (2.23) is equivalent to the equation (2.24) which is invariant under the transformations (2.17). This explains how it is possible to achieve the two conditions (2.19), (2.20) with the help of a single function $\beta(k, l)$. After imposing (2.19), (2.20), the ambiguity due to rescaling (2.16) is fixed up to an overall rescaling with $\beta(k, l) = \text{const}$.

2.3 Formal solution

Let us introduce the following combinations of the coefficients

$$h(k, l) = f(k+1, l)g(k, l), \quad H(k, l) = F(k, l+1)G(k, l), \quad (2.26)$$

that remain invariant under the redefinitions (2.16) and satisfy the following two equations as a consequence of Eqs. (2.15)

$$\begin{aligned} \frac{h(k, l-1)}{h(k, l)} &= \frac{k-l+1}{k-l+2} \frac{d+k+l-2}{d+k+l-3}, \\ \frac{H(k+1, l)}{H(k, l)} &= \frac{d+k+l-2}{d+k+l-1} \frac{k-l+1}{k-l+2}. \end{aligned} \quad (2.27)$$

The condition (2.14) gives

$$\begin{aligned} h(k, l) \frac{d+2k}{d+2k-2} - h(k-1, l) - H(k, l) \frac{d+2l-2}{(k-l)(d+k+l-3)} - \lambda^2 &= 0, \\ h(k, l) \frac{d+2k}{(k-l+2)(d+k+l-3)} + H(k, l) \frac{d+2l-2}{d+2l-4} - H(k, l-1) - \lambda^2 &= 0. \end{aligned} \quad (2.28)$$

Given $H(k, l)$ and $h(k, l)$, σ is reconstructed by (2.19), (2.20), (2.26)

$$f(k+1, l) = \sqrt{n(k, l) \frac{(d+k-2)k}{k+1} h(k, l)}, \quad k > 0, \quad l \leq k, \quad (2.29)$$

$$g(k, l) = \text{sign}(h(k, l)) \sqrt{\frac{k+1}{k(d+k-2)n(k, l)} h(k, l)}, \quad k > 0, \quad l \leq k, \quad (2.30)$$

$$f(1, 0) = \sqrt{n(0, 0)h(0, 0)}, \quad g(0, 0) = \text{sign}(h(0, 0)) \sqrt{h(0, 0)/n(0, 0)}, \quad (2.31)$$

$$F(k, l+1) = \sqrt{N(k, l) \frac{(d+l-3)(k-l+1)}{k-l} H(k, l)}, \quad k > l \geq 0, \quad (2.32)$$

$$G(k, l) = \text{sign}(H(k, l)) \sqrt{\frac{k-l}{(d+l-3)(k-l+1)N(k, l)}} H(k, l), \quad k > l \geq 0. \quad (2.33)$$

The formal general solution of the equations (2.27) and (2.28) is

$$\begin{aligned} h(k, l) = & -\frac{(d+2l_0-2)(k_0-k)(d+k_0+k-1)}{(d+2k)(k-l+1)(d+k+l-2)} H(k_0, l_0) + \\ & + \frac{(d+2k_0)(k-l_0+1)(d+k+l_0-2)}{(d+2k)(k-l+1)(d+k+l-2)} h(k_0, l_0) + \\ & - \lambda^2 \frac{(k_0-k)(d+k_0+k-1)(k-l_0+1)(d+k+l_0-2)}{(d+2k)(k-l+1)(d+k+l-2)}, \end{aligned} \quad (2.34)$$

$$\begin{aligned} H(k, l) = & \frac{(d+2k_0)(l_0-l)(d+l_0+l-3)}{(d+2l-2)(d+k+l-2)(k-l+1)} h(k_0, l_0) + \\ & + \frac{(d+k_0+l-2)(k_0-l+1)(d+2l_0-2)}{(d+k+l-2)(k-l+1)(d+2l-2)} H(k_0, l_0) + \\ & - \lambda^2 \frac{(d+k_0+l-2)(k_0-l+1)(d+l_0+l-3)(l_0-l)}{(d+k+l-2)(k-l+1)(d+2l-2)}. \end{aligned} \quad (2.35)$$

From Eqs. (2.34), (2.35) we observe that $H(k, l)$ and $h(k, l)$ with $k \geq l \geq 0$ are determined by their values $H(k_0, l_0)$ and $h(k_0, l_0)$ at any point (k_0, l_0) with $k_0 \geq l_0 \geq 0$. That this should have happen follows from formal consistency of the “finite difference equations” (2.27), (2.28) in the integral variables k and l . Note that denominators in (2.34), (2.35) are nonzero for $k \geq l$, which is the condition that the second row of a Young diagram is not longer than the first one.

An important property of the solution (2.34), (2.35) is expressed by

Lemma. If $H(p, s) = 0$ or $h(s-1, p) = 0$ for some p and s then $H(k, s) = 0 \quad \forall \quad k$ and $h(s-1, l) = 0 \quad \forall \quad l$

$$H(p, s) = 0 \quad \text{or} \quad h(s-1, p) = 0 : \Rightarrow \quad H(k, s) = h(s-1, l) = 0 \quad \forall \quad k, l. \quad (2.36)$$

Proof. Suppose that $H(p, s) = 0$. Let us use (2.34), (2.35) to express all $h(k, l)$ and $H(k, l)$ in terms of $H(p, s)$ and $h(p, s)$ setting $k_0 = p$ and $l_0 = s$. The first term in (2.34) is then proportional to $H(p, s)$ and hence vanishes. The other terms contain a factor of $(k-s+1)$ that is zero for $k = s-1$. Hence $h(s-1, l) = 0$. The second term in (2.35) is proportional to $H(p, s)$ and hence vanishes. The other terms contain a factor of $(s-l)$ which implies that $H(k, s) = 0$. That (2.36) is true if $h(s-1, p) = 0$ is proved analogously. \square

The solution (2.34), (2.35) is formal since it does not take into account (2.10) which demands

$$H(n, n) = h(n-1, n) = 0. \quad (2.37)$$

This can be treated as a boundary condition for Eqs. (2.27)-(2.28). Generally, (2.37) is not true for (2.34), (2.35). It turns out that if we would use the formula (2.34), (2.35) for all h and H except for those given by (2.37), the compatibility condition (2.11) will not be satisfied only in the sector of rectangular Young diagrams

$$(D^2 + \sigma^2) |_{W \neq 0}, \quad (D^2 + \sigma^2) |_{V/W=0}, \quad (2.38)$$

where \mathcal{W} denotes the set of operators that map rectangular Young diagrams to rectangular ones. For example, $\sigma_-^1(n, n-1)\sigma_-^2(n, n) \in \mathcal{W}$ because it maps $\mathbf{Y}(n, n)$ to $\mathbf{Y}(n-1, n-1)$.

In the general case, the equations (2.1) contain $W^{a(k), b(l)}$ with all $k \geq l$. $H(k, l)$ and $h(k, l)$ are defined by the two-parametric solution (2.34), (2.35). In this case the boundary condition (2.37) is not satisfied. Hence the curvatures (2.1) are inconsistent for generic coefficients $H(k, l)$ and $h(k, l)$ that should be chosen appropriately to describe a spin s field.

2.4 Scaling gauges

The coefficients $h(k, l)$ (2.34) and $H(k, l)$ (2.35) can be chosen to be real. In order the coefficients $f(k, l)$, $g(k, l)$, $F(k, l)$ and $G(k, l)$ also be real we demand

$$\text{sign}(n(k, l)) = \text{sign}(h(k, l)), \quad \text{sign}(N(k, l)) = \text{sign}(H(k, l)) \quad (2.39)$$

in (2.19), (2.20). The signs in (2.29), (2.32) are chosen so that $f(k, l)$ and $F(k, l)$ are positive.

In the practical analysis we will deal with differential forms of degrees 0 and 1 and use the two ways of fixation of the field rescaling ambiguity. The *type-I scaling gauge* is

$$n_{(0)}(k, l) = n_{(1)}(k, l) = \text{sign}(h(k, l)), \quad N_{(0)}(k, l) = N_{(1)}(k, l) = \text{sign}(H(k, l)), \quad (2.40)$$

where the subscripts (0) or (1) indicate a degree of the differential form in question. Although the type-I scaling is most convenient for the general analysis, it is inapplicable to the analysis of the flat massless limit where it should be replaced by the *type-II scaling gauge*

$$N_{(1)}(k, l) = \text{sign}(H(k, l)) \left| \frac{1}{m^2 + \lambda^2(s-l-1)(s+l+d-4)} \right|, \quad n_{(1)}(k, l) = \text{sign}(h(k, l)), \quad (2.41)$$

$$N_{(0)}(k, l) = \text{sign}(H(k, l)), \quad n_{(0)}(k, l) = \text{sign}(h(k, l)) \left| \frac{1}{m^2 + \lambda^2(s-k-2)(s+k+d-3)} \right|. \quad (2.42)$$

It is easy to check that (2.41) and (2.42) respect the condition (2.25). Since $N_{(1)}(k, l)$ is k -independent and $n_{(0)}(k, l)$ is l independent, we introduce notation

$$N_{(1)}(l) = N_{(1)}(k, l), \quad n_{(0)}(k) = n_{(0)}(k, l). \quad (2.43)$$

Note that

$$N_{(1)}(l) = -n_{(0)}(l-1). \quad (2.44)$$

2.5 Spin s systems

It turns out that solutions of (2.34), (2.35), that satisfy (2.36) with some s , correspond to a spin s field.

From the condition $h(s-1, l) = 0$ and (2.26) it follows that either $f(s, l) = 0$ or $g(s-1, l) = 0$. Consider the case where $f(s, l) = g(s-1, l) = 0$ for all l . It is easy to see that $f(s, l) = 0$ leads to $\sigma_-^1(s, l) = 0$, which means that the connections $W^{a(u), b(v)}$ with $u \geq s$ do not contribute to the curvatures $R^{a(k), b(l)}$ (2.4) with $k \leq s-1$. Analogously, $g(s-1, l) = 0$ leads to $\sigma_+^1(s-1, l) = 0$, which means in turn that $W^{a(k), b(l)}$ with $k \leq s-1$ do not contribute

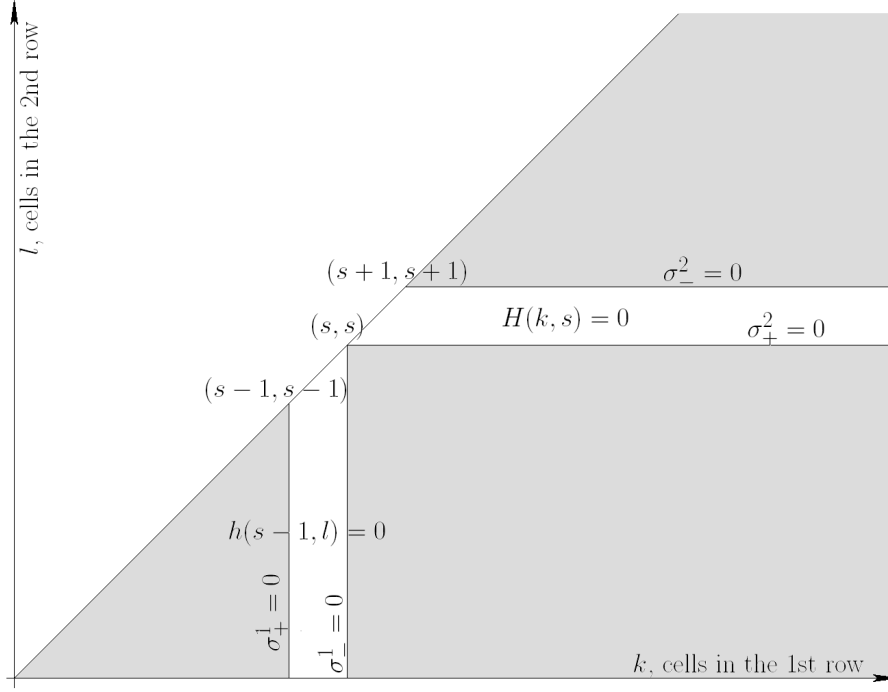


Figure 1: Spin s field, non-critical $m \neq m_t$. Fields in distinct shaded regions can form independent unfolded systems (2.4)

to the curvatures $R^{a(u),b(v)}$ (2.4) with $u \geq s$. So, if both $f(s, l) = 0$ and $g(s-1, l) = 0$ are satisfied the respective curvatures do not mix $W^{a(u),b(v)}$ with $u \geq s$ and $W^{a(k),b(l)}$ with $k \leq s-1$. Hence, in this case $W^{a(k),b(l)}$ with $k \leq s-1$ (finite triangle region in Fig. 1) and $W^{a(u),b(v)}$ with $u \geq s$ (both infinite triangular and infinite rectangular regions in Fig. 1) form independent subsystems. Relaxing the conditions $f(s, l) = 0$ or $g(s-1, l) = 0$ leads to the mixing of the fields $W^{a(k),b(l)}$ with $k \leq s-1$ and $W^{a(u),b(v)}$ with $u \geq s$.

The condition $H(k, s) = 0$ can be analyzed analogously. It is easy to see that in the case where $f(s, l) = g(s-1, l) = F(k, s+1) = G(k, s) = 0$ for some s and all k and l , the shaded regions in Fig. 1 form independent subsystems that are not mixed in the curvatures (2.4).

The subsystem associated to the infinite rectangular region in Fig. 1 contains $W^{a(k),b(l)}$ with $k \geq s$, $l \leq s$. In the case of 0-forms ($p = 0$), it describes a so-called Weyl module of a massive particle of spin s that encodes nontrivial degrees of freedom of the system. The corresponding coefficients $H(k, l)$ and $h(k, l)$ are

$$\begin{aligned} h(k, l) &= -\frac{(m^2 + \lambda^2(s-k-2)(s+k+d-3))(k-s+1)(d+s+k-2)}{(d+2k)(k-l+1)(d+k+l-2)}, \\ H(k, l) &= -\frac{(m^2 + \lambda^2(s-l-1)(s+l+d-4))(s-l)(d+s+l-3)}{(d+2l-2)(k-l+1)(d+k+l-2)}, \end{aligned} \quad (2.45)$$

where the free parameter m identifies with the mass as one can see by the direct analysis of the equations (2.4) for 0-forms $C^{a(k),b(l)}$ with $k \geq s$, $l \leq s$ analogous to the analysis in the 1-form sector carried out in Section 3.3.2. Note that Eq. (2.45) provides an alternative parametrization of the general solution (2.34), (2.35) with s interpreted as an arbitrary (not

necessarily integer) parameter.

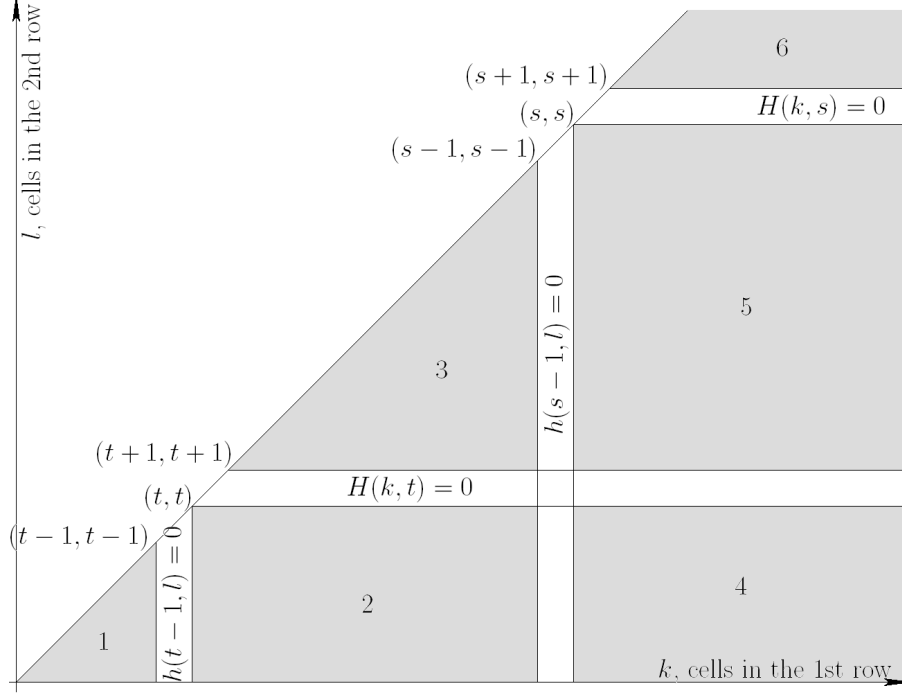


Figure 2: Partially massless field, $m = m_t$. The set of fields can be divided into six groups

Both finite ($k \leq s-1$) and infinite ($l > s$) triangle regions in Fig. 1 do not respect (2.37). Nevertheless, in Section 3 we will show that the subsystem associated to the finite triangle plays important role in the description of massive fields, leading to consistent unfolded equations (2.1) upon introducing an appropriate mixture between 0- and 1-forms.

It is easy to see that for special values $m = m_t$ (t is integer) in (2.45)

$$m_t^2 = -\lambda^2(s-t-1)(d+s+t-4), \quad t = 0, 1, \dots, s-1 \quad (2.46)$$

the following property takes place

$$H(k, t) = h(t-1, l) = 0. \quad (2.47)$$

Analogously to the condition (2.36), Eq. (2.47) subdivides the system into further subsystems as shown in Fig. 2. Since the region 4 does not contain fields valued in $\mathbf{Y}(n, n)$ the condition (2.37) here is inessential. The regions 2 and 5 both respect (2.37) as a consequence of (2.36) and (2.46). To summarize, the regions 2, 4 and 5 respect (2.11) leading to consistent curvatures, while in the regions 1, 3 and 6 the condition (2.37) is not respected.

The critical values of mass (2.46) correspond to partially massless [31, 13, 21] and massless [18, 19] fields for $t < s-1$ and $t = s-1$, respectively. Namely, the subsystem associated to the region 2 of Fig. 2 was used to describe partially massless fields in terms of 1-forms in [21]. The region 5 describes the corresponding Weyl module.

Note that letting $t, s \in \mathbb{R}$ in Eqs. (2.45), (2.46) provides an alternative parametrization of the general solution (2.34), (2.35).

Also, from (2.45) it is easy to see that if m^2 is given by (2.46) with integer t , $t > s$ and $\lambda^2 \neq 0$ then the Weyl module of such a field satisfies (2.47). In this case the Weyl module (the infinite rectangular region in Fig. 1) admits a submodule. It consists of 0-forms $C^{a(k),b(l)}$ with $s \leq k \leq t-1$. This phenomenon was observed for the case of massive spin 0 field in [28]. Such degeneracies usually indicate the coexistence of different dual descriptions of the same field theoretical system, and hence may be interesting from various perspectives.

The following comment is now in order. It is well-known (see e.g. [7] and references therein) that any consistent unfolded equation for a set of forms of a definite degree implies that this set spans a module of the global symmetry of the system, which is Poincaré or $(A)dS$ symmetry in the case of most interest in this paper. That the compatibility conditions are not satisfied only at the boundary of the area of definition of Young tableaux suggests the following interpretation. Gauge fields of massless and partially massless fields are described by finite dimensional tensor $o(d-1, 2)$ -modules T [19, 21]. In this case, the parameter of mass is discrete, being associated to the length of the second row of the respective Young tableaux [21]. These finite dimensional modules can be understood as particular members of the set of modules V_m parametrized by the continuous parameter of mass m . More precisely, for the specific values m_i of m associated to massless and partially massless fields, V_{m_i} acquire infinite dimensional submodules U_{m_i} such that $V_{m_i}/U_{m_i} = T_i$ are the finite dimensional modules used for the description of (partially) massless fields. That the unfolded equations are compatible almost everywhere except for the boundary of the region associated to the Young tableaux (i.e. finite dimensional representations of the Lorentz group) confirms this picture. Most likely, the obstruction to the continuation of the construction at the boundary of the Young tableaux region, shown in this section for generic values of mass, signals that the corresponding infinite dimensional module V_m does not decompose into a sum of finite dimensional modules of the Lorentz algebra as is also anticipated for the submodules U_{m_i} . In other words, although the module V_m exists, it may admit no interpretation in the standard field-theoretical terms of finite-component Lorentz fields.

3 Unfolded equations for massive fields of any spin

3.1 Field pattern

Unfolded equations (2.4) formulated in terms of 0-forms $C^{a(k),b(l)}$ with $k \geq s$ and $l \leq s$ (infinite rectangular region in Fig. 1) are consistent. They describe the Weyl module of a massive HS field, that operates with gauge invariant combinations of gauge fields. To figure out how to introduce gauge potentials, it is useful to use the observation of [13, 22] that the Lagrangian of a spin s massive field can be represented as a sum of the Lagrangians for the set of spin s' , $0 \leq s' \leq s$ massless fields supplemented with certain mass-dependent lower-derivative terms that mix massless fields of different spins. Hence, it is natural to expect that the unfolded formulation of a massive spin s particle can be given in terms of fields of the unfolded formulation for the set of massless fields of spins $0 \leq s' \leq s$.

As shown in [19, 20], a spin s' symmetric massless field is described by the set of 1-form connections valued in traceless tensors of $o(d-1, 1)$ that have symmetries of two-row Young

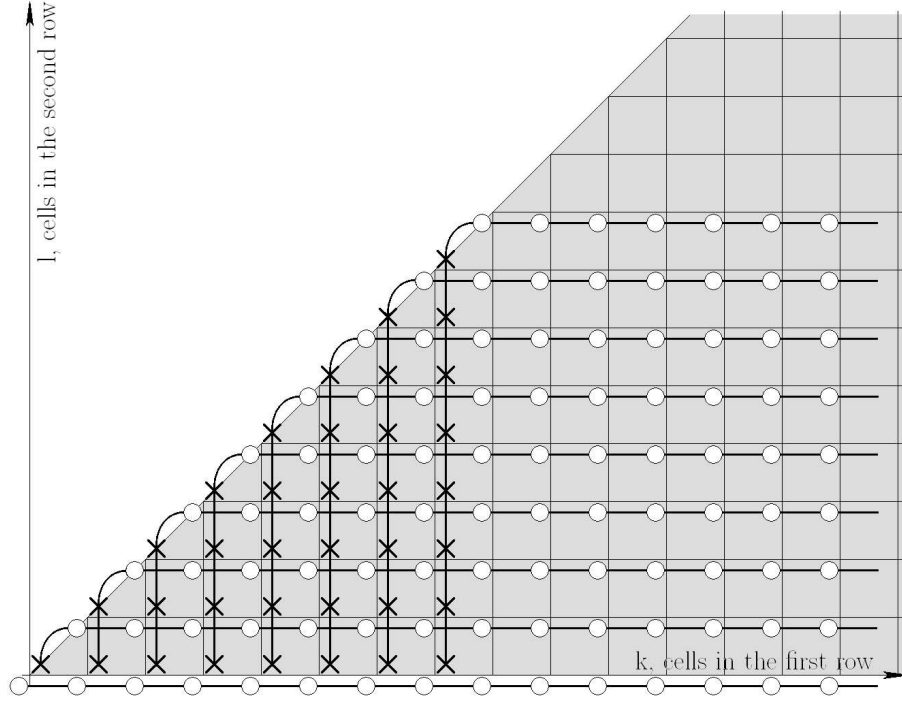


Figure 3: On this plot the set of fields used for description of a massive spin s field is shown for $s = 8$. “ \times ” denote 1-forms and “ \circ ” denote 0-forms. The fields associated to each massless field are connected by a solid line. The set of fields of a massive spin s model is a combination of the sets of fields of the massless unfolded systems of spins $s' = 0, \dots, s$.

tableaux with $s' - 1$ cells in the first row and $0 \leq l \leq s' - 1$ cells in the second row

$$\text{1-forms: } \begin{array}{|c|} \hline s' - 1 \\ \hline l \\ \hline \end{array}, \quad 0 \leq l \leq s' - 1$$

and Weyl 0-forms valued in traceless tensors of $o(d - 1, 1)$ possessing symmetries of two-row Young tableaux with $k \geq s'$ cells in the first row and s' cells in the second row

$$\text{0-forms: } \begin{array}{|c|} \hline k \\ \hline s' \\ \hline \end{array}, \quad k \geq s'.$$

Thus the full set of fields appropriate for the description of a spin s massive field should contain 1-form connections valued in $\mathbf{Y}(k, l)$ with $k \leq s - 1$, $l \leq k$ and 0-forms valued in $\mathbf{Y}(k, l)$ with $l \leq s$, $k \geq l$. We observe that the resulting set of 1-forms precisely corresponds to the triangular region of Fig. 3 while the set of 0-forms corresponds to the unification of the infinite rectangular region with the triangular region of Fig. 3. In this section it will be shown how the compatibility problems in the triangular region encountered in the case of forms of definite degree are resolved by a mixture between 1-forms and 0-forms of this region.

Note that the gluing the Weyl module to the combined set of 1- and 0-forms in the triangle is not necessary for formal consistency but is needed to relate the 1-forms to the propagating degrees of freedom described by the 0-form Weyl module associated to the infinite rectangular region.

3.2 Unfolded equations via mixing of forms of different degrees

General linear unfolded equations (1.2), (1.3) formulated in terms of $(p+1)$ -forms $W^{a(k),b(l)}$ and p -forms $C^{a(k),b(l)}$, that possess symmetries of two-row Young tableaux, have the structure

$$\begin{aligned} 0 = R_{\{p+2\}}^{a(k),b(l)} &= dW^{a(k),b(l)} + G_{(p+1)}^{a(k),b(l)}(W, C) = dW + \sigma_{(p+1)}W + \kappa C, \\ 0 = R_{\{p+1\}}^{a(k),b(l)} &= dC^{a(k),b(l)} + G_{(p)}^{a(k),b(l)}(W, C) = dC + \sigma_{(p)}C + \alpha W. \end{aligned} \quad (3.1)$$

A label in parentheses refers to the differential form degree of a curvature. Since α carries differential form degree 0, it is independent of the vielbein and does not change a tensor type, *i.e.*,

$$\alpha \mathbf{Y}(k, l) = \alpha(k, l) \mathbf{Y}(k, l), \quad (3.2)$$

where $\alpha(k, l)$ are some coefficients. (Here we use the shorthand notation $\mathbf{Y}(k, l)$ for a tensor described by the Young tableau with two rows of lengths $k \geq l$.)

The operators κ are built from two vielbeins, carrying the differential form degree two and contain several terms

$$\kappa = \kappa_0 + \kappa_{--}^{12} + \kappa_{++}^{12} + \kappa_{+-}^{12} + \kappa_{-+}^{12}, \quad (3.3)$$

$$\begin{aligned} \kappa_0 : \mathbf{Y}(k, l) &\rightarrow \mathbf{Y}(k, l), \quad \kappa_{--}^{12} : \mathbf{Y}(k, l) \rightarrow \mathbf{Y}(k-1, l-1), \\ \kappa_{++}^{12} : \mathbf{Y}(k, l) &\rightarrow \mathbf{Y}(k+1, l+1), \quad \kappa_{-+}^{12} : \mathbf{Y}(k, l) \rightarrow \mathbf{Y}(k-1, l+1), \\ \kappa_{+-}^{12} : \mathbf{Y}(k, l) &\rightarrow \mathbf{Y}(k+1, l-1). \end{aligned} \quad (3.4)$$

Each of these operators except for κ_0 is determined by the Young symmetry properties up to a (k, l) -dependent coefficient. κ_0 contains two linearly independent terms, each defined up to a (k, l) -dependent coefficient.

The operators $\sigma_{(p+1)}$ and $\sigma_{(p)}$ act on $(p+1)$ -forms and p -forms, respectively and add one differential form degree. They still have the form (2.5)-(2.8) but the compatibility conditions (1.4) are modified because of presence of κ and α . Hence the respective coefficients $F_{(p+1)}(k, l)$ ($F_{(p)}(k, l)$), $G_{(p+1)}(k, l)$ ($G_{(p)}(k, l)$), $f_{(p+1)}(k, l)$ ($f_{(p)}(k, l)$) and $g_{(p+1)}(k, l)$ ($g_{(p)}(k, l)$) may differ from those given in (2.26) and (2.34), (2.35).

In the new framework, the compatibility conditions imply

$$\alpha \sigma_{(p+1)} = \sigma_{(p)} \alpha, \quad (3.5)$$

$$\alpha \kappa = D^2 + (\sigma_{(p)})^2, \quad (3.6)$$

$$\kappa \alpha = D^2 + (\sigma_{(p+1)})^2, \quad (3.7)$$

$$\sigma_{(p+1)} \kappa = \kappa \sigma_{(p)}. \quad (3.8)$$

Eqs. (3.6), (3.7) express κ in terms of σ .

From (3.5) it follows that

$$h_{(p+1)}(k, l) = h_{(p)}(k, l) \quad \text{if} \quad \alpha(k, l) \alpha(k+1, l) \neq 0, \quad (3.9)$$

$$H_{(p+1)}(k, l) = H_{(p)}(k, l) \quad \text{if} \quad \alpha(k, l) \alpha(k, l+1) \neq 0. \quad (3.10)$$

Indeed, let us for example derive (3.9). The operators $\sigma_-^1(k+1, l)$ and $\sigma_+^1(k, l)$ for p - and $(p+1)$ -forms are related by the conditions (3.5)

$$\begin{aligned}\alpha(k, l)\sigma_{(p+1)-}^1(k+1, l) &= \sigma_{(p)-}^1(k+1, l)\alpha(k+1, l), \\ \alpha(k+1, l)\sigma_{(p+1)+}^1(k, l) &= \sigma_{(p)+}^1(k, l)\alpha(k, l).\end{aligned}$$

This implies

$$\begin{aligned}\alpha(k, l)f_{(p+1)}(k+1, l) &= f_{(p)}(k+1, l)\alpha(k+1, l), \\ \alpha(k+1, l)g_{(p+1)}(k, l) &= g_{(p)}(k, l)\alpha(k, l).\end{aligned}\tag{3.11}$$

Multiplying these relations and using (2.26) we obtain (3.9).

The gauge transformations that follow from (1.5) are

$$\delta W = d\xi + \sigma_{(p+1)}\xi, \quad \delta C = -\alpha\xi.\tag{3.12}$$

In addition to the independent scaling redefinitions (2.16) for p - and $(p+1)$ -forms, the equations (3.1) possess the ambiguity in the field redefinitions

$$W = \tilde{W} + \tilde{\sigma}C\tag{3.13}$$

that leads to following redefinition of the operators

$$\sigma_{(p)} \rightarrow \sigma_{(p)} + \alpha\tilde{\sigma}, \quad \sigma_{(p+1)} \rightarrow \sigma_{(p+1)} + \tilde{\sigma}\alpha, \quad \kappa \rightarrow \kappa + \tilde{\sigma}\sigma_{(p)} + \sigma_{(p+1)}\tilde{\sigma} + \tilde{\sigma}\alpha\tilde{\sigma}.\tag{3.14}$$

It is not hard to see that to single out a solution of the compatibility conditions that describes a spin s particle in the case of $p = 0$, the following condition, that plays a role analogous to that of (2.36), should be imposed

$$\alpha(k, l) = 0 \quad \text{for } k \geq s, \quad \alpha(k, l) \neq 0 \quad \text{for } k \leq s-1.\tag{3.15}$$

Then from (3.5) it follows that

$$\sigma_{(0)+}^1(s-1, l) = \sigma_{(1)-}^1(s, l) = 0.\tag{3.16}$$

Indeed, the condition (3.15) and the second of the conditions (3.16) imply that 1-forms $W^{a(u), b(v)}$ with $u \geq s$ do not contribute to the equations for all 0-forms and 1-forms $W^{a(k), b(l)}$ with $k \leq s-1$. Hence, (3.15) guarantees that the 1-forms in the infinite triangular and rectangular regions of Fig. 3 decouple from all other fields and can be dropped from the system. Since the curvatures for 0-forms valued in $\mathbf{Y}(u, v)$ with $u \geq s$ do not contain 1-forms, the analysis of Section 2 is applicable to this case. The first of the relations (3.16) implies that (2.36) is satisfied. This means that the 0-forms in the infinite triangular region of Fig. 1 also decouple and can be dropped from the system. As a result we are left with the set of fields of Fig. 3.

Let us now consider the 1- and 0-forms in the triangular region of Fig. 3. These should be mixed by the equations (3.1) to resolve the compatibility problem of the unmixed case of Section 2. Together with the Weyl module these fields should lead to an unfolded system

that consistently describes massive HS particles. To ensure that the Weyl module is glued to the sector of mixed 0- and 1-forms we demand

$$\sigma_{(0)-}^1(s, l) \neq 0. \quad (3.17)$$

The coefficients $f_{(0)}(s, l)$ will be specified later.

Note that (3.14) implies that any $\sigma_{(0)}$ (or $\sigma_{(1)}$ related to $\sigma_{(0)}$ by (3.5)) can be obtained by some field redefinition (3.13) in the sector where 1- and 0-forms are mixed. Our goal is to fix this freedom in such a way that, for particular values of m and λ that correspond to massless and partially massless fields, the unfolded system decompose into appropriate subsystems. The solution

$$H_{(0)}(k, l) = H_{(1)}(k, l) = H(k, l), \quad h_{(0)}(k, l) = h_{(1)}(k, l) = h(k, l) \quad (3.18)$$

with H and h given by (2.45) satisfies this requirement. The compatibility problems encountered in Section 2 are resolved by the nontrivial mixture of 1- and 0-forms. Recall that H and h (2.45) solve (2.11) on its restriction to $\bar{\mathcal{W}}$, while the restriction of $D^2 + \sigma^2$ to \mathcal{W} is nonzero (2.38). Taking into account (3.6), we see that with this choice of H and h we should set $\kappa = 0$ if $\kappa \in \bar{\mathcal{W}}$, while $\kappa|_{\mathcal{W}}$ should be non-zero.

To summarize, the unfolded equations that describe the Weyl module in terms of 0-forms $C^{a(k), b(l)}$ with $k \geq s$, $l \leq s$ are

$$R_{\{1\}} = DC + \sigma_{(0)}C = 0 \quad \text{for} \quad R_{\{1\}}^{a(k), b(l)} \quad \text{with} \quad k \geq s, \quad l \leq s. \quad (3.19)$$

These fields belong to the infinite rectangular region of Fig. 3. The finite triangle region contains 1-forms $W^{a(k), b(l)}$ and 0-forms $C^{a(k), b(l)}$ with $k \leq s - 1$ that satisfy the equations

$$R_{\{2\}} = DW + \sigma_{(1)}W = 0, \quad \text{for} \quad R_{\{2\}}^{a(k), b(l)} \quad \text{with} \quad l < k \leq s - 1, \quad (3.20)$$

$$R_{\{2\}} = DW + \sigma_{(1)}W + \kappa C = 0, \quad \text{for} \quad R_{\{2\}}^{a(n), b(n)} \quad \text{with} \quad n \leq s - 1, \quad (3.21)$$

$$R_{\{1\}} = DC + \sigma_{(0)}C + \alpha W = 0 \quad \text{for} \quad R_{\{1\}}^{a(k), b(l)} \quad \text{with} \quad k \leq s - 1. \quad (3.22)$$

Both $\sigma_{(0)}$ and $\sigma_{(1)}$ are still defined by (2.26) and (2.45) up to the rescaling ambiguity (2.17) (independently for 0- and 1-forms). Explicit formulae for the coefficients, that satisfy the normalization conditions (2.19), (2.20) with arbitrary N and n (2.21) that respect (2.25), are given in Appendix B.

For the type-I scaling gauge (2.40), $f_{(0)}(s, l)$ is determined by (2.15) up to an overall factor which can be fixed in such a way that

$$f_{(0)}(s, l) = \sqrt{\frac{1}{(s-l)(d+s+l-3)}}. \quad (3.23)$$

Since in the type-I scaling gauge $\sigma_{(0)} = \sigma_{(1)}$ by (2.40), (3.18), we obtain from (3.5) $\alpha(k, l) = \alpha(0, 0)$. By the leftover scaling symmetry we can set

$$\alpha(k, l) = 1. \quad (3.24)$$

For the type-II scaling gauge (2.41), (2.42), $f_{(0)}(s, l)$ is still defined by (3.23) while the expression for $\alpha(k, l)$ in terms of $\alpha(0, 0)$ acquires the form

$$\alpha(k, l) = \alpha(0, 0) \sqrt{\left| \frac{N_{(1)}(0)}{\prod_{i=l}^k N_{(1)}(i)} \right|}.$$

The leftover scaling symmetry is used to fix $\alpha(0, 0) \sqrt{|N_{(1)}(0)|} = 1$, so that the final expression for α is

$$\alpha(k, l) = \frac{1}{\sqrt{\left| \prod_{i=l}^k N_{(1)}(i) \right|}}. \quad (3.25)$$

The explicit form of κ (3.4) is

$$\begin{aligned} \kappa_{--}^{12}(n, n) &= \frac{1}{\alpha(n, n)} \sigma_{(1)-}^1(n, n-1) \sigma_{(1)-}^2(n, n), \\ \kappa_{++}^{12}(n, n) &= \frac{1}{\alpha(n, n)} \sigma_{(1)+}^2(n+1, n) \sigma_{(1)+}^1(n, n), \\ \kappa_0(n, n) &= \frac{1}{\alpha(n, n)} (D^2 + \sigma_{(1)-}^1(n+1, n) \sigma_{(1)+}^1(n, n) + \sigma_{(1)+}^2(n, n-1) \sigma_{(1)-}^2(n, n)), \end{aligned} \quad (3.26)$$

where D^2 is given in Eq. (2.3). Other components of κ in (3.3), (3.4) are zero. Note that the condition $\kappa_{--}^{12}(s, s) \neq 0$ is required by (3.17) rather than by the compatibility conditions.

The gauge transformations for 0- and 1-forms in the triangular region of Fig. 3 are given by (3.12) with $p = 0$, while the Weyl 0-forms in the rectangular region are gauge invariant.

3.3 Dynamical fields and Singh-Hagen system

To show that the constructed unfolded system with $m \neq m_t$ (the case of partially massless fields is special and will be considered in Section 3.5) indeed describes a massive HS field, it is convenient to use the σ_- cohomology analysis [28, 7] with the σ_- complex defined precisely as in the massless case $m = 0$ for the set of massless fields of spins $0, 1, \dots, s$. This implies that the σ_- cohomological grade for 0(1)-forms identifies with the length of the first (second) row of their Young diagrams. The 1-form fields of the lowest grade described by one-row diagrams are called frame-like fields.

Since so defined σ_- is independent of m , such analysis treats m^2 as a deformation parameter. Hence it treats constraints like $L\phi_1 + m^2\phi_2 = 0$ as dynamical equation as they are in the massless case of $m^2 = 0$, rather than constraints that express ϕ_2 via ϕ_1 . Such constraints should therefore be taken into account after performing the σ_- cohomology analysis to show that the dynamical field $\phi_{a(s)}$ satisfies the equations (1.1) while the other fields are either gauged away or expressed in terms of dynamical field by the equations of motion.

The σ_- cohomology analysis for a massless field gives the following results (see e.g. [7]). Dynamical fields for every massless field of spin $k+1$ contained in the frame-like field $W^{a(k)}$ consist of two irreducible symmetric tensors

$$W^{a(k)} \rightarrow \boxed{k+1} \oplus \boxed{k-1}, \quad 0 \leq k \leq s-1. \quad (3.27)$$

A massless spin zero field is described by the scalar 0-form C . This set of fields coincides with that proposed by Zinoviev in [13].

The leftover gauge symmetry is:

$$\delta(W(k, 0))^{a(k)} = D(\xi(k, 0))^{a(k)} + (\sigma_{(1)-}^1 \xi(k+1, 0))^{a(k)} + (\sigma_{(1)+}^1 \xi(k-1, 0))^{a(k)}. \quad (3.28)$$

For generic values of m , the first component in (3.27) of each frame-like field except for $W^{a(s-1)}$ can be gauge fixed to zero by the gauge parameter $\xi^{a(k+1)}$ using the second term on the r.h.s. of (3.28). The remaining gauge parameter ξ is used to gauge away the 0-form C . After the gauge symmetry is completely fixed, the remaining non-zero field projections are

$$W^{a(k)} \rightarrow \boxed{k-1}, \quad 0 < k < s-1, \quad (3.29)$$

$$W^{a(s-1)} \rightarrow \boxed{s-2} \oplus \boxed{s}. \quad (3.30)$$

This is the set of fields of Singh and Hagen [10]. All other fields involved in the unfolded formulation either are pure gauge or are expressed via derivatives of the Singh-Hagen fields.

Note that for special values of the mass m the second term in the transformation law (3.28) may degenerate not allowing to gauge fix to zero the highest component field in (3.27) for one or another k . This corresponds to the cases of massless and partially massless fields considered in more detail in Section 3.5 while in this section we assume that this does not happen.

3.3.1 Vanishing of lower-spin supplementary fields

Let us show that, in the chosen gauge, all fields (3.29), (3.30) except for traceless symmetric projection of $W^{a(s-1)}$ vanish by virtue of the equations of motion. We shall use two equations for each frame-like field. The first one expresses the Lorentz-like auxiliary field via the frame-like field. The second one imposes a non-trivial condition on the frame-like field. For the lower-spin supplementary fields (i.e. those, associated to the massless spin s' fields with $s' < s$) this condition equates them to zero. For the dynamical field associated to the massless spin s field it imposes the dynamical equation.

The proof will be given by induction. Namely, assuming that $W^a = 0$ and $W^{a,b} = 0$ we prove that

$$W^{a(l)} = 0, \quad W^{a(l),b} = 0 \quad \forall l < k \quad \implies \quad W^{a(k)} = 0, \quad W^{a(k),b} = 0. \quad (3.31)$$

Let us first consider the special cases of $s' = 0, 1, 2$. Using the gauge conditions

$$C(0, 0) = 0, \quad W(0, 0) = 0, \quad (3.32)$$

the lowest equation in the spin zero sector

$$0 = DC(0, 0) + \sigma_{(0)-}^1 C(1, 0) + \alpha(0, 0)W(0, 0)$$

leads to

$$C^a = 0. \quad (3.33)$$

The only non-zero projection of W^a is its trace $W_{\rho|\rho}$ since, according to (3.29), all other its components have been gauged fixed to zero. From the trace of the equation

$$0 = D(C(1, 0))^a + (\sigma_{(0)-}^1 C(2, 0))^a + (\sigma_{(0)+}^1 C(0, 0))^a + (\sigma_{(0)-}^2 C(1, 1))^a + \alpha(1, 0)(W(1, 0))^a \quad (3.34)$$

we obtain using (3.32) that $W_{\rho|\rho} = 0$ and hence

$$W^a = 0. \quad (3.35)$$

Also it follows from (3.34) that

$$C^{a,b} = 0. \quad (3.36)$$

To show that $W^{a,b} = 0$, we observe that $W_{\mu|}^{a,b}$ has three irreducible components

$$W^{a,b} \rightarrow \begin{array}{|c|c|} \hline & \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \\ \hline \end{array}. \quad (3.37)$$

Projecting the equation

$$0 = D(W(1, 0))^a + (\sigma_{(1)-}^1 W(2, 0))^a + (\sigma_{(1)+}^1 W(1, 0))^a + (\sigma_{(1)-}^2 W(1, 1))^a \quad (3.38)$$

to the symmetry types of the first two terms in (3.37) we obtain that the corresponding projections of $W^{a,b}$ are zero. Taking into account (3.36), the trace of the equation

$$0 = D(C(1, 1))^{a,b} + (\sigma_{(0)-}^1 C(2, 1))^{a,b} + (\sigma_{(0)+}^2 C(1, 0))^{a,b} + \alpha(1, 1)(W(1, 1))^{a,b}$$

implies that the third component of $W^{a,b}$ in (3.37) is also zero. Hence,

$$W^{a,b} = 0. \quad (3.39)$$

The equations (3.35), (3.39) provide the first step of induction. Let us now prove (3.31). The projection of the equation

$$\begin{aligned} 0 = & D(W(k-1, 0))^{a(k-1)} + (\sigma_{(1)-}^1 W(k, 0))^{a(k-1)} + \\ & + (\sigma_{(1)+}^1 W(k-2, 0))^{a(k-1)} + (\sigma_{(1)-}^2 W(k-1, 1))^{a(k-1)} \end{aligned} \quad (3.40)$$

to the symmetry type of $\begin{array}{|c|} \hline k-1 \\ \hline \end{array}$ implies that the second component of $W^{a(k)}$ in (3.27) is zero. Since the first component has been gauge fixed to zero, we obtain

$$W^{a(k)} = 0. \quad (3.41)$$

The situation with $W^{a(k),b}$ is a bit more involved. It has the following irreducible components

$$W^{a(k),b} \rightarrow \begin{array}{|c|c|} \hline & k \\ \hline \end{array} \oplus \begin{array}{|c|} \hline k-1 \\ \hline \end{array} \oplus \begin{array}{|c|} \hline k+1 \\ \hline \end{array} \oplus \begin{array}{|c|} \hline k \\ \hline \end{array} \oplus \begin{array}{|c|} \hline k \\ \hline \end{array}.$$

The component $\begin{array}{|c|} \hline k \\ \hline \end{array}$ is pure gauge and can be gauge fixed to zero as mentioned in the beginning of Section 3.3. The projections of the equation

$$0 = D(W(k, 0))^{a(k)} + (\sigma_{(1)-}^1 W(k+1, 0))^{a(k)} + (\sigma_{(1)+}^1 W(k-1, 0))^{a(k)} + (\sigma_{(1)-}^2 W(k, 1))^{a(k)} \quad (3.42)$$

to the symmetry types

$$\begin{array}{|c|} \hline k-1 \\ \hline \square \end{array}, \quad \begin{array}{|c|} \hline k \\ \hline \square \end{array}, \quad \begin{array}{|c|} \hline k+1 \\ \hline \square \end{array}$$

imply that the components of $W^{a(k),b}$ of these symmetry types are zero. The remaining component $\begin{array}{|c|} \hline k \\ \hline \square \end{array}$ vanishes by virtue of the rank k symmetric part of the equation

$$0 = D(W(k-1, 1))^{a(k-1),b} + (\sigma_{(1)-}^1 W(k, 1))^{a(k-1),b} + (\sigma_{(1)+}^1 W(k-2, 1))^{a(k-1),b} + (\sigma_{(1)-}^2 W(k-1, 2))^{a(k-1),b} + (\sigma_{(1)+}^2 W(k-1, 0))^{a(k-1),b}. \quad (3.43)$$

Hence,

$$W^{a(k),b} = 0. \quad (3.44)$$

Thereby, all fields $W^{a(k)}$ and $W^{a(k),b}$ with $k < s-1$ are shown to be zero.

3.3.2 Dynamical equations

Let us now derive the dynamical equation in Minkowski and $(A)dS$ cases. Projection of (3.40) with $k = s-1$ to $\begin{array}{|c|} \hline s-2 \\ \hline \square \end{array}$ implies that $W_{\rho}^{\rho a(s-2)} = 0$, while the traceless totally symmetric part $\begin{array}{|c|} \hline s \\ \hline \square \end{array}$ of $W^{a(s-1)}$ remains non-zero and cannot be gauged away. This is the spin s dynamical field denoted $\phi_{a(s)}$. It is totally symmetric and traceless

$$\phi_{a(s)} = W_{a|a(s-1)}, \quad \phi^{\rho}_{\rho a(s-2)} = 0. \quad (3.45)$$

The projection of (3.42) to $\begin{array}{|c|} \hline k+1 \\ \hline \square \end{array}$ expresses $W^{a(k),b}$ in terms of $dW^{a(k)}$ which is not zero only for $k = s-1$. Analogously to Subsection 3.2.2, we can use (3.42) and (3.43) with $k = s-1$ to prove that $\begin{array}{|c|} \hline s \\ \hline \square \end{array}$ is the only non-vanishing projection of $W^{a(s-1),b}$. One consequence of this result is that $W^{a(s-1),b}$ is traceless

$$W^{\rho|}_{a(s-1),\rho} = 0, \quad W^{\rho|}_{a(s-2)\rho,b} = 0. \quad (3.46)$$

With the help of (3.45) and (3.46), the contraction of indices a and μ of the equation

$$0 = R_{\mu\nu|a(s-1)} = D_{\mu}\phi_{\nu a(s-1)} - D_{\nu}\phi_{\mu a(s-1)} + F_{(1)}(s-1, 1)(W_{\nu|a(s-1),\mu} - W_{\mu|a(s-1),\nu}) \quad (3.47)$$

yields

$$D^{\rho}\phi_{\rho a(s-1)} = 0. \quad (3.48)$$

The projection of the equation $R_{\{2\}}^{a(s-1),b} = 0$ to the symmetry type $\begin{array}{|c|} \hline s \\ \hline \square \end{array}$ gives

$$0 = D^{\rho}W_{a|a(s-1),\rho} + G_{(1)}(s-1, 0)(d-2)\phi_{a(s)}. \quad (3.49)$$

Expressing $W_{a|a(s-1),\rho}$ in terms of ϕ by means of (3.47) we obtain

$$D^{\rho}W_{a|a(s-1),\rho} = \frac{1}{F_{(1)}(s-1, 1)}(D^{\rho}D_a\phi_{\rho a(s-1)} - D^{\rho}D_{\rho}\phi_{a(s)}). \quad (3.50)$$

Using Eqs. (2.3), (3.48) and substituting (3.50) into (3.49) we obtain

$$0 = D^\rho D_\rho \phi_{a(s)} - (\lambda^2(d+s-2) + (d-2)H(s-1,0)) \phi_{a(s)}, \quad (3.51)$$

where $H(s-1,0)$ is defined in (2.26). Using (2.45) we finally obtain

$$0 = D^\rho D_\rho \phi_{a(s)} + (\lambda^2(s^2 + s(d-6) - 2d + 6) + m^2) \phi_{a(s)}. \quad (3.52)$$

This is the field equation for a massive spin s field in the $(A)dS$ background. The mass-like term reproduces that obtained in [32]. In the flat case $\lambda = 0$ we recover the equation (1.1).

It is easy to see that, the only fields that remain non-zero on shell in the chosen gauge are the 0-forms that constitute the Weyl module and the projections of the 1-forms $W^{a(s-1),b(l)}$

to $\begin{array}{|c|} \hline s \\ \hline l \\ \hline \end{array}$. All of them are expressed by the unfolded equations via higher derivatives of the dynamical field $\phi_{a(s)}$.

3.4 Spin 2 example

Let us consider the example of a spin 2 field of generic mass, $m \neq m_t$ (2.46) for $t = 0, 1$. In this case, the full set of fields consists of 1-forms W , W^a , $W^{a,b}$, 0-forms C , C^a , $C^{a,b}$ and 0-forms $C^{a(k),b(l)}$, $k \geq 2 \geq l$ of the Weyl module. Unfolded equations (3.20)-(3.22) are

$$0 = DC + f_{(0)}(1,0)e_m C^m + \alpha(0,0)W, \quad (3.53)$$

$$0 = DC^a + f_{(0)}(2,0)e_m C^{am} + g_{(0)}(0,0)e^a C + F_{(0)}(1,1)e_m C^{a,m} + \alpha(1,0)W^a, \quad (3.54)$$

$$0 = DC^{a,b} + f_{(0)}(2,1)e_m (C^{am,b} + \frac{1}{2}C^{ab,m}) + G_{(0)}(1,0)(e^b C^a - e^a C^b) + \alpha(1,1)W^{a,b}, \quad (3.55)$$

$$0 = DW + f_{(1)}(1,0)e_m W^m + \alpha^{-1}(0,0)f_{(0)}(1,0)F_{(0)}(1,1)e_m e_n C^{m,n}, \quad (3.56)$$

$$0 = DW^a + g_{(1)}(0,0)e^a W + F_{(1)}(1,1)e_m W^{a,m}, \quad (3.57)$$

$$\begin{aligned} 0 = & DW^{a,b} + G_{(1)}(1,0)(e^b W^a - e^a W^b) + 2\alpha^{-1}(1,1)G_{(0)}(1,0)g_{(0)}(0,0)e^b e^a C + \\ & + \alpha^{-1}(1,1)(\lambda^2 + G_{(0)}(1,0)F_{(0)}(1,1))(e^a e_m C^{m,b} + e^b e_m C^{a,m}) + \\ & + \alpha^{-1}(1,1)f_{(0)}(2,1)F_{(0)}(2,2)e_n e_m C^{an,bm} \end{aligned} \quad (3.58)$$

plus equations for the 0-forms in the Weyl module that follow from (3.53)-(3.58). The coefficients in curvatures both for 0- and 1-forms satisfy (2.45), (2.19), (2.20). The gauge symmetries (3.12) are

$$\delta C = -\alpha(0,0)\xi, \quad \delta C^a = -\alpha(1,0)\xi^a, \quad \delta C^{a,b} = -\alpha(1,1)\xi^{a,b}, \quad (3.59)$$

$$\delta W = D\xi + f_{(1)}(1,0)e_m \xi^m, \quad (3.60)$$

$$\delta W^a = D\xi^a + g_{(1)}(0,0)e^a \xi + F_{(1)}(1,1)e_m \xi^{a,m}, \quad (3.61)$$

$$\delta W^{a,b} = D\xi^{a,b} + G_{(1)}(1,0)(e^b \xi^a - e^a \xi^b). \quad (3.62)$$

Analogously to the massless spin 2 case, using (3.61), one fixes $\xi^{a,b}$ by requiring the anti-symmetric part of W^a to be zero. Then we can fix ξ and ξ^a setting $C = 0$ and $W = 0$. In

this gauge Eq. (3.53) implies $C^m = 0$. Then from (3.54) it follows that $W_{\rho|}{}^\rho = 0$. Eq. (3.56) states that $C^{a,b} = 0$. Contracting one fiber and one space-time index of (3.55) we obtain that $W_{\rho|}{}^{\rho,a} = 0$. Then from (3.57) it is easy to obtain that the dynamical field $\phi_{aa} = W_{a|a}$ is transversal. Eq. (3.57) expresses $W^{a,b}$ in terms of derivatives of ϕ_{aa} . Plugging this expression into (3.58), contracting one fiber and one space-time index and symmetrizing the others we obtain the spin 2 equation in AdS_d :

$$D^\rho D_\rho \phi_{aa} + (-2\lambda^2 + m^2)\phi_{aa} = 0. \quad (3.63)$$

3.5 Massless and partially massless fields

Let us now discuss peculiarities of the massless and partially massless cases. Using the type-II scaling gauge, it will be shown that a massive spin s field decomposes into the set of massless fields of spins s' with $0 \leq s' \leq s$ at $m = 0$, $\lambda^2 = 0$ and into one spin s massless field and one spin $s - 1$ massive field at $m = 0$, $\lambda^2 \neq 0$. Analogously, for $m = m_t$ a massive spin s field will be shown to decompose into one partially massless field of spin s and depth t and one spin t massive field.

As discussed in Section 2, massless and partially massless fields are characterized by the special values of mass m_t (2.46) for which the condition (2.47) $H(k, t) = 0$ holds. As follows from (2.26) this can be achieved by setting $F(k, t + 1) = 0$ and/or $G(k, t) = 0$. Analogously, $h(t - 1, l) = 0$ can be achieved by setting $f(t, l) = 0$ and/or $g(t - 1, l) = 0$. In the type-I scaling gauge, from (2.40), (2.29)-(2.33) we have

$$\begin{aligned} F_{(0)}(k, t + 1) &= G_{(0)}(k, t) = f_{(0)}(t, l) = g_{(0)}(t - 1, l) = \\ &= F_{(1)}(k, t + 1) = G_{(1)}(k, t) = f_{(1)}(t, l) = g_{(1)}(t - 1, l) = 0. \end{aligned} \quad (3.64)$$

In the type-II scaling gauge we have from (2.41), (2.42), (2.29)-(2.33)

$$\begin{aligned} F_{(0)}(k, t + 1) &= G_{(0)}(k, t) = g_{(0)}(t - 1, l) = 0, \quad f_{(0)}(t, l) \neq 0, \\ G_{(1)}(k, t) &= f_{(1)}(t, l) = g_{(1)}(t - 1, l) = 0, \quad F_{(1)}(k, t + 1) \neq 0. \end{aligned} \quad (3.65)$$

The main difference between the two scalings is that the type-II scaling gauge keeps non-zero $\sigma_{(0)-}^1$ and $\sigma_{(1)-}^2$ which is crucial for the field equations to remain sensible for the special values of the parameter of mass.

Indeed, for $\lambda^2 = 0$, the Weyl module of a spin s' massless field consists of 0-forms valued in $\mathbf{Y}(k, s')$, $k \geq s'$. In Minkowski space, the unfolded massless equations have the form [20]

$$R_{\{1\}}(k, s') = 0 = dC(k, s') + \sigma_{(0)-}^1 C(k + 1, s'), \quad k \geq s', \quad (3.66)$$

i.e., $\sigma_{(0)+}^1 = \sigma_{(0)+}^2 = \sigma_{(0)-}^2 = 0$ and consequently $H_{(0)}(k, l) = h_{(0)}(k, l) = 0$. It is easy to see from (2.45) that both $H_{(0)}(k, l)$ and $h_{(0)}(k, l)$ of the massive Weyl module indeed vanish at $\lambda^2 = 0$ and $m^2 = 0$. The choice of $n_{(0)}(k, l)$ in (2.42) provides non-zero $\sigma_{(0)-}^1$ in the flat massless limit. $H_{(0)}(k, l) = 0$ leads to $\sigma_{(0)+}^2 = \sigma_{(0)-}^2 = 0$, *i.e.*, Eq. (3.66) is reproduced and the massive Weyl module decomposes into s sets of 0-forms valued in $\mathbf{Y}(k, s')$ with definite $s' < s$ and arbitrary $k \geq s$ for each set. Compared to the Weyl module of a massive particle, the combined set of the Weyl modules for massless fields of spins s' , $0 \leq s' \leq s$ contains

additional 0-forms that form the finite triangular set in Fig. 3. Let us discuss the origin of this difference in more detail.

At $\lambda^2 = 0$, $m^2 = 0$ Eq. (2.45) yields $H(k, l) = h(k, l) = 0$ both for 0- and for 1-forms of the triangular region of Fig. 3. In the scaling (2.41), (2.42) this leads to vanishing of all σ except for $\sigma_{(1)-}^2$ and $\sigma_{(0)-}^1$. As a result, the triangular set of 1-forms decomposes into vertical lines, each containing $W^{a(s'-1), b(l)}$ with $l \leq s' - 1$ and definite $s' < s$, while the triangular set of 0-forms decomposes into horizontal lines, each containing $C^{a(k), b(s')}$ with $k \geq s'$ and definite $s' < s$. Since from (3.25) it follows that $\alpha = 0$ at $m^2 = \lambda^2 = 0$, the curvatures for 0-forms of the triangular region do not contain 1-forms. As a result, these 0-forms form the completion of the Weyl modules of the decoupled massless particles. It can be shown that κ_{++}^{12} and κ_0 also vanish. κ_{--}^{12} remains non-zero and glues $W^{a(s'-1), b(s'-1)}$ to $C^{a(s'), b(s')}$. This describes the standard gluing of gauge potentials to the Weyl module in the massless case.

In the case of $\lambda^2 \neq 0$, a massless field is a particular case of $t = s - 1$ of partially massless fields that allows us to use Fig. 2 to visualize the decomposition of the Weyl module in the massless limit in $(A)dS$. So all 0-forms of the region 5 have s cells in the second row, while the 0-forms of the region 2 have $s - 1$ cells in the first row. The Weyl module of a spin s massive particle consists of 0-forms of regions 4 and 5.

The $(A)dS$ Weyl module of a spin s massless field consists of 0-forms valued in $\mathbf{Y}(k, s)$, $k \geq s$ forming region 5 of Fig. 2. The unfolded equations are

$$R_{\{1\}}(k, s) = 0 = dC(k, s) + \sigma_{(0)-}^1 C(k + 1, s) + \sigma_{(0)+}^1 C(k - 1, s), \quad k \geq s.$$

Decoupling of 0-forms of the Weyl module associated to a massless spin s field at $\lambda^2 \neq 0$ requires $H_{(0)}(k, s - 1) = 0$. This is indeed true for (2.45) with $m^2 = 0$. The remaining 0-forms, valued in $\mathbf{Y}(k, l)$ with $k \geq s$, $l < s$, that form the region 4 of Fig. 2, constitute a part of the Weyl module of a spin $s - 1$ massive field with the specific mass m_s (2.46) (recall that this value of mass fulfils the condition $H_{(0)}(k, s) = h_{(0)}(s - 1, l) = 0$ of the decomposition of the massive Weyl module into the two parts forming regions 2 and 4 of Fig. 2).

The lacking part of the Weyl module of the spin $s - 1$ massive field contains 0-forms valued in $\mathbf{Y}(s - 1, l)$ with any $l \leq s - 1$ (i.e. region 2 in Fig. 2). In the case of $\lambda^2 \neq 0$, the 1-forms $W^{a(s-1), b(l)}$ of a massless spin s field decouple from the rest 1-forms, being still glued to the massless spin s Weyl module. Since from (3.25) it follows that $\alpha(s - 1, l) = 0$ for $m^2 = 0$, the curvatures for 0-forms $C^{a(s-1), b(l)}$ do not contain 1-forms. These 0-forms form a lacking part of the Weyl module of the massive spin $s - 1$ particle. 0- and 1-forms valued in $\mathbf{Y}(k, l)$ with $k < s - 1$ are still mixed and form a finite triangle in Fig. 3 of a massive spin $s - 1$ particle.

Let us briefly consider the partially massless case. At $m = m_t$, the condition (2.47) is satisfied, thereby (3.65) is fulfilled for the type-II scaling gauge. It is easy to see from (3.25) that

$$\alpha(k, l) = 0, \quad l \leq t \leq k \quad (3.67)$$

i.e., $\alpha(k, l) = 0$ for (k, l) from the region 2 of Fig. 2. It follows from (3.65), (3.67) that unfolded equations do not mix fields of the following two sets.

The first set consists of mixed 1- and 0-forms of the region 1 glued to 0-forms of regions 2 and 4 of Fig. 2. These fields describe a massive spin t field of mass m_s (2.46).

The second set consists of 1-forms of the region 2. Since $\sigma_{(1)-}^2(k, t+1) \neq 0$ by (3.65), these 1-forms are glued to mixed 1- and 0-forms valued in the region 3. At the same time, 0- and 1-forms of the region 3 are glued to the 0-forms of the region 5 because $\sigma_{(0)-}^1(s, l) \neq 0$.

Compared to the approach of [21] this set provides a *non-minimal* description of a partially massless field of spin s and depth t that contains additional fields along with constraints that express them algebraically in terms of derivatives of the dynamical fields modulo Stueckelberg degrees of freedom that are gauged away by additional Stueckelberg gauge symmetries. Indeed, to show that the two descriptions are equivalent we observe that the 0-forms of the region 3 can be gauge fixed to zero by the gauge transformation (3.12)

$$C^{a(k), b(l)} = 0, \quad k \leq s-1, \quad l \geq t+1.$$

Then, equating curvatures (3.22) for 0-forms of the region 3 to zero we obtain $W^{a(k), b(l)} = 0$ for $k \leq s-2$, $l \geq t+1$, while $W^{a(s-1), b(l)}$ with $l \geq t+1$ are expressed in terms of 0-forms that belong to the partially massless Weyl module

$$\sigma_{(0)-}^1(s, l)C(s, l) + \alpha(s-1, l)W(s-1, l) = 0, \quad l \geq t+1. \quad (3.68)$$

Taking into account (3.68) for $l = t+1$, one finds that the term $\sigma_{(1)-}^2(s-1, t+1)W(s-1, t+1)$ in the curvature $R_{\{2\}}^{a(s-1), b(t)}$ glues the partially massless Weyl module to the gauge potentials just in the same way as in the standard description where the gauge potentials of the region 2 are glued to the Weyl module via² [33]

$$R_{\{2\}}^{a(s-1), b(t)} = e_c e_d C^{a(s-1)c, b(t)d}.$$

Thereby the non-minimal description of a partially massless field reduces to the standard one. Let us note that, although the type-II scaling gauge is most convenient for the analysis of the massive field decoupling in the partially massless limit, the standard description of a single partially massless field simplifies in the type-I scaling gauge.

Let us now analyze the dynamical content of the unfolded field equations in the partially massless case where $\sigma_{(1)-}^1(t, l) = \sigma_{(1)+}^1(t-1, l) = 0$ by (3.65) and 1-forms are valued in $\mathbf{Y}(k, l)$ with $t \leq k \leq s-1$. Analogously to the massive case we use the σ_- cohomology analysis as in the massless case for the set of massless fields of spins $t+1, t+2, \dots, s$. As a result, we are left with a set of frame-like fields

$$W^{a(k)} \rightarrow \boxed{k+1} \oplus \boxed{k-1}, \quad t \leq k \leq s-1 \quad (3.69)$$

(cf (3.27)). The leftover gauge symmetry is still given by (3.28) for k : $t \leq k \leq s-1$. The first component in (3.69) of each frame-like field except for $W^{a(s-1)}$ can be gauged away by means of a gauge parameter $\xi^{a(k+1)}$:

$$\mathcal{P}_{\text{Tr}=0}(W^{a|a(k)}) = 0. \quad (3.70)$$

Here $\mathcal{P}_{\text{Tr}=0}$ denotes the projector to the traceless part. The only gauge parameter that cannot be fixed this way is $\xi^{a(t)}$. This is the parameter of a differential gauge symmetry.

²We acknowledge a useful discussion with E. Skvortsov of the specificities of gluing the Weyl module in the partially massless case.

The gauge transformation with the parameter $\xi^{a(t)}$ must be accompanied by the gauge transformations with the parameters $\xi^{a(k)}$ with $t < k \leq s-1$ to satisfy the gauge fixing condition (3.70). Indeed, the gauge transformation of $\mathcal{P}_{\text{Tr}=0}(W^{a|a(t)})$ is

$$\mathcal{P}_{\text{Tr}=0}(\delta W^{a|a(t)}) = \mathcal{P}_{\text{Tr}=0}(D^a \xi^{a(t)}) + F_{(1)}(t+1, 0) \xi^{a(t+1)}$$

and hence

$$\xi^{a(t+1)} = -\frac{\mathcal{P}_{\text{Tr}=0}(D^a \xi^{a(t)})}{F_{(1)}(t+1, 0)}.$$

Analogously, one can show that

$$\xi^{a(k)} = (-1)^{k-t} \frac{\mathcal{P}_{\text{Tr}=0}(\overbrace{D^a \dots D^a}^{k-t} \xi^{a(t)})}{\prod_{i=t+1}^k F_{(1)}(i, 0)}. \quad (3.71)$$

After the Stueckelberg gauge symmetry is completely fixed, we are left with non-zero field projections

$$\begin{aligned} W^{a(k)} &\rightarrow \boxed{k-1}, \quad t \leq k < s-1, \\ W^{a(s-1)} &\rightarrow \boxed{s-2} \oplus \boxed{s} \end{aligned} \quad (3.72)$$

and the differential gauge symmetry (3.28) with gauge parameters satisfying (3.71).

The only non-zero projection of $W^{a(t)}$ is

$$\psi^{a(t-1)} = W_{\rho|}^{\rho a(t-1)}, \quad \delta \psi^{a(t-1)} = D_{\rho} \xi^{\rho a(t-1)}. \quad (3.73)$$

It can neither be expressed in terms of other fields nor gauged away by a Stueckelberg gauge transformation, hence being a dynamical field. It follows from

$$0 = D(W(t, 0))^{a(t)} + (\sigma_{(1)-}^1 W(t+1, 0))^{a(t)} + (\sigma_{(1)-}^2 W(t, 1))^{a(t)} \quad (3.74)$$

that the only non-zero projections of $W^{a(t),b}$ are $\boxed{t-1}$ and \boxed{t} . The hook-type projection is expressed in terms of derivative of ψ by (3.74). Also Eq. (3.74) expresses the projection of $W^{a(t),b}$ to \boxed{t} , namely $W_{\rho|}^{a(t),\rho}$, in terms of $D\psi$ and $W_{\rho|}^{\rho a(t)}$. Since there are no other restrictions on the fields $W_{\rho|}^{a(t),\rho}$ and $W_{\rho|}^{\rho a(t)}$, one of them, say $W_{\rho|}^{\rho a(t)}$, remains unrestricted and should be identified with the second dynamical field

$$\varphi^{a(t)} = W_{\rho|}^{\rho a(t)}. \quad (3.75)$$

Its transformation law follows from (3.28), (3.71)

$$\begin{aligned} \delta \varphi^{a(t)} &= g_{(1)}(t, 0) \frac{(d+2t)(d+t-2)}{d+2t-2} \xi^{a(t)} - \\ &- \frac{(d+2t-2)D_{\rho}D^{\rho}\xi^{a(t)} + t(d+2t-4)D_{\rho}D^a\xi^{\rho a(t-1)} - t(t-1)D^{\rho}D^{\lambda}\xi_{\rho\lambda}^{a(t-2)}\eta^{aa}}{(t+1)(d+2t-2)F_{(1)}(t+1, 0)}. \end{aligned}$$

The further analysis of equations for partially massless case is analogous to the massive case. $\phi^{a(s)} = \mathcal{P}_{\text{Tr}=0}(W^{a|a(s-1)})$ is the third dynamical field. From (3.28), (3.71) it follows that

$$\delta\phi^{a(s)} = (-1)^{s-t-1} \frac{\mathcal{P}_{\text{Tr}=0}(\overbrace{D^a \dots D^a}^{s-t} \xi^{a(t)})}{\prod_{i=t+1}^{s-1} F_{(1)}(i, 0)}. \quad (3.76)$$

This is in accordance with the known property that gauge transformations of partially massless fields contain higher derivatives [31, 13, 21].

4 Action

The advantage of the unfolded formulation is that it operates with gauge invariant curvatures (1.2), (3.19)-(3.22) which can be used to construct gauge invariant actions. Any action, bilinear in these curvatures, is manifestly gauge invariant. The gauge invariant action that describes a unitary theory should satisfy the conditions

$$\frac{\delta S}{\delta W^{a(k), b(l)}} \equiv 0, \quad l \geq 2, \quad \frac{\delta S}{\delta C^{a(q), b(r)}} \equiv 0, \quad q \geq 2, \quad (4.1)$$

which generalize *the extra field decoupling condition* for massless [18, 19] and partially massless [21] fields to the massive case. This is equivalent to the condition that the action is free of derivatives higher than two. Another requirement is *the massless decomposition condition* which demands that the part of the action that contains two derivatives of the dynamical fields do not mix dynamical fields associated to different massless fields. Here we refer to the fact, that in agreement with construction of Zinoviev [13], the massless flat limit of a massive action should give the sum of free massless actions for spins from 0 to s . In this manner, $W^{a(k-1), b(l)}$ is associated to the l -th derivative of a spin k dynamical massless field. Hence, such fields should not contribute to the action if $l \geq 2$. Analogously, in the massless case, the Weyl 0-form $C^{a(q), b(r)}$ contains q derivatives of a spin $(q - r)$ dynamical field. Thereby such fields with $q \geq 2$ also are not allowed to contribute to a unitary action. The extra field decoupling condition (4.1) along with the massless decomposition condition fix the action up to an overall factor and total derivatives, guaranteeing that it contains at most two derivatives of the dynamical fields and has correct structure in the massless limit.

In this section we discard the contribution of the 0-forms from the Weyl module to the curvatures (3.21), (3.22) using that this does not violate gauge invariance because the Weyl tensors are gauge invariant. In practice, this only applies to the curvatures $R_{\{2\}}^{a(s-1), b(s-1)}$ and $R_{\{1\}}^{a(s-1), b(s-1)}$ because other curvatures that appear in this section do not contain 0-forms that belong to the Weyl module.

Most of the consideration of this section applies to an arbitrary scaling gauge (2.19), (2.20), unless the conditions (2.41), (2.42) are explicitly referred to in the analysis of the flat massless limit.

4.1 Action derivation

To begin with, let us find the part of the action $S_{1,1}$ built from the curvatures for 1-forms, that satisfies (4.1) up to C -dependent terms. To compensate the C -dependent terms we will

then add appropriate terms $S_{0,0}$ built from 0-forms so that

$$S = S_{1,1} + S_{0,0}$$

will satisfy (4.1), yielding correct field equations.

$S_{1,1}$ is constructed in terms of the curvatures $R_{\{2\}}$ analogously to the frame-like actions for massless [18, 19] and partially massless [21] fields

$$S_{1,1} = \sum_{1 \leq l \leq k \leq s-1} a_{k,l} \langle R_{\{2\}}^{a(k),b(l)} | R_{\{2\}}^{a(k),b(l)} \rangle, \quad (4.2)$$

where $\langle \dots | \dots \rangle$ is the inner product (2.18) and the coefficients $a_{k,l}$ remain to be determined.

Using (2.21), the part of the variation of the action, that contains $R_{\{2\}}$ is

$$\begin{aligned} \delta S_{1,1} = -2 \sum_{1 \leq l \leq k \leq s-1} & \left(\langle \delta W(k, l) | \sigma_{(1)-}^1 R_{\{2\}}(k+1, l) \rangle \left(\frac{1}{n_{(1)}(k, l)} a_{k+1, l} - a_{k, l} \right) + \right. \\ & + \langle \delta W(k, l) | \sigma_{(1)-}^2 R_{\{2\}}(k, l+1) \rangle \left(\frac{1}{N_{(1)}(k, l)} a_{k, l+1} - a_{k, l} \right) + \\ & + \langle \delta W(k, l) | \sigma_{(1)+}^1 R_{\{2\}}(k-1, l) \rangle (n_{(1)}(k-1, l) a_{k-1, l} - a_{k, l}) + \\ & \left. + \langle \delta W(k, l) | \sigma_{(1)+}^2 R_{\{2\}}(k, l-1) \rangle (N_{(1)}(k, l-1) a_{k, l-1} - a_{k, l}) \right). \quad (4.3) \end{aligned}$$

Obviously, the extra field decoupling condition (4.1) demands

$$a_{k+1, l} = a_{k, l} n_{(1)}(k, l), \quad a_{k, l+1} = a_{k, l} N_{(1)}(k, l). \quad (4.4)$$

Using (2.40) this gives

$$a_{k, l} = z(k, l) a_{s-1, 1} = z(k, l) a \quad (4.5)$$

for the type-I scaling gauge. Here $z(k, l)$ is a sign factor

$$z(k, l) = \prod_{i=1}^{l-1} \text{sign}(H(s-1, i)) \prod_{j=k}^{s-2} \text{sign}(h(j, l)).$$

For the type-II scaling gauge (2.41), (2.42) one obtains

$$a_{k, l} = z(k, l) a_{s-1, 1} \prod_{n=1}^{l-1} |N_{(1)}(n)|. \quad (4.6)$$

As explained in more detail in Section 4.3, to obtain non-zero variation in the flat massless limit we should set

$$a_{k, l} = z(k, l) a \prod_{n=0}^{l-1} |N_{(1)}(n)|, \quad (4.7)$$

where a is both m^2 and λ^2 independent.

The variation (4.3) remains nonzero because the coefficients $a_{k,l}$ are zero at $k \leq 0$. As a result,

$$\delta S = 2 \sum_{1 \leq k \leq s-1} a_{k,1} \left(\langle \sigma_{(1)+}^2 \delta W(k, 0) | R_{\{2\}}(k, 1) \rangle + \langle \delta W(k, 1) | \sigma_{(1)+}^2 R_{\{2\}}(k, 0) \rangle \right). \quad (4.8)$$

Thus the extra field decoupling condition determines $S_{1,1}$ up to an overall factor a . The only terms in (4.8) that contain two derivatives of the dynamical fields are

$$\langle \sigma_{(1)+}^2 \delta W(k, 0) | DW(k, 1) \rangle \quad \text{and} \quad \langle \delta W(k, 1) | \sigma_{(1)+}^2 DW(k, 0) \rangle.$$

Since they do not mix fields associated to different massless fields, the massless decomposition condition is also fulfilled.

Additional terms in the variation result from the presence of 0-forms in the curvatures $R_{\{2\}}^{a(n),b(n)}$ (3.21) valued in the rectangular Young diagrams, *i.e.*, from the terms $\sum_n a_{n,n} \langle R_{\{2\}}^{a(n),b(n)} | R_{\{2\}}^{a(n),b(n)} \rangle$. These give

$$\begin{aligned} \Delta(\delta S_{1,1}) = & 2 \left(\sum_{1 \leq n \leq s-1} (a_{n,n} \langle \delta W(n, n) | \kappa_0 R_{\{1\}}(n, n) \rangle - a_{n,n} \langle \delta C(n, n) | \kappa_0 R_{\{2\}}(n, n) \rangle) + \right. \\ & \sum_{1 \leq n \leq s-1} (a_{n,n} \langle \delta W(n, n) | \kappa_{++}^{12} R_{\{1\}}(n-1, n-1) \rangle - \\ & - a_{n-1,n-1} \frac{\alpha(n, n)}{\alpha(n-1, n-1)} \langle \delta C(n-1, n-1) | \kappa_{--}^{12} R_{\{2\}}(n, n) \rangle) + \\ & + \sum_{1 \leq n \leq s-2} (a_{n,n} \langle \delta W(n, n) | \kappa_{--}^{12} R_{\{1\}}(n+1, n+1) \rangle - \\ & \left. - a_{n+1,n+1} \frac{\alpha(n, n)}{\alpha(n+1, n+1)} \langle \delta C(n+1, n+1) | \kappa_{++}^{12} R_{\{2\}}(n, n) \rangle) \right), \end{aligned} \quad (4.9)$$

where $\alpha(k, l)$ is determined by (3.24) for the type-I scaling and by (3.25) for the type-II scaling. Note that the terms

$$\langle \delta W(s-1, s-1) | \kappa_{--}^{12} R_{\{1\}}(s, s) \rangle \quad \text{and} \quad \langle \delta C(s, s) | \kappa_{++}^{12} R_{\{2\}}(s-1, s-1) \rangle$$

do not appear in (4.9) because the term $\kappa_{--} C^{a(s),b(s)}$, constructed by means of the 0-form that belongs to the Weyl module, have been dropped from the curvature $R_{\{2\}}^{a(s-1),b(s-1)}$.

The terms (4.9) can be compensated by terms bilinear in the curvatures $R_{\{1\}}$ which have the following general form

$$\begin{aligned} S_{0,0} = & \sum_{k,l} \pi_{k,l} \int \epsilon_{l_1 \dots l_d} R_{\{1\}}^{l_1 a(k-1), b(l)} R_{\{1\}}^{l_2 a(k-1), b(l)} e^{l_3} \dots e^{l_d} + \\ & + \sum_{k,l} \varrho_{k,l} \int \epsilon_{l_1 \dots l_d} R_{\{1\}}^{a(k), l_1 b(l-1)} R_{\{1\}}^{l_2 a(k), b(l-1)} e^{l_3} \dots e^{l_d} + \\ & + \sum_{k,l} \tau_{k,l} \int \epsilon_{l_1 \dots l_d} R_{\{1\}}^{l_1 a(k), l_2 b(l)} R_{\{1\}}^{l_3 a(k), b(l)} e^{l_4} \dots e^{l_d} + \\ & + \sum_{k,l} \varphi_{k,l} \int \epsilon_{l_1 \dots l_d} R_{\{1\}}^{l_1 a(k-1), b(l)} R_{\{1\}}^{l_2 a(k-1), b(l)} e^{l_3} \dots e^{l_d}. \end{aligned} \quad (4.10)$$

It is useful to observe that almost all terms in (4.10) can be re-written in terms of the inner product $\langle \dots | \dots \rangle$ and some differential form of degree 2 built from the background vielbein. For example, the third term in (4.10) is proportional to

$$\langle R_{\{1\}}(k+1, l+1) | \sigma_{(1)+}^2 \sigma_{(1)+}^1 R_{\{1\}}(k, l) \rangle. \quad (4.11)$$

The analysis of terms in this form is much easier than in the general form (4.10) because their variation can be easily found by means of Bianchi identities and conjugation properties (2.21). Such a representation works nicely provided that the curvatures carry enough indices. For example, to vary the term

$$\langle \sigma_{(1)+}^2 \delta W(k, l) | \sigma_{(1)-}^1 \sigma_{(1)+}^1 R_{\{1\}}(k, l+1) \rangle$$

it is enough to use (2.21) for σ_+^2 to obtain

$$-\frac{1}{N_{(1)}(k, l)} \langle \delta W(k, l) | \sigma_{(1)-}^2 \sigma_{(1)-}^1 \sigma_{(1)+}^1 R_{\{1\}}(k, l+1) \rangle.$$

Unfortunately, such a representation does not work in the case of $l = 0$ where (2.21) is inapplicable and the variation of such terms should be analyzed separately. Let these special terms be denoted as S_{spec} , while the other regular terms be denoted as S_{reg} , so that

$$S_{0,0} = S_{spec} + S_{reg}. \quad (4.12)$$

It can be easily checked that the variation of

$$\begin{aligned} S_{reg} = & - \sum_{reg} \left\langle \frac{a}{\alpha} R_{\{1\}} | \kappa R_{\{1\}} \right\rangle = - \sum_{n=2}^{s-1} \left(\frac{a_{n-1, n-1}}{\alpha(n-1, n-1)} \langle R_{\{1\}}(n-1, n-1) | \kappa_{--}^{12} R_{\{1\}}(n, n) \rangle + \right. \\ & \left. + \frac{a_{n, n}}{\alpha(n, n)} \langle R_{\{1\}}(n, n) | \kappa_{++}^{12} R_{\{1\}}(n-1, n-1) \rangle + \frac{a_{n, n}}{\alpha(n, n)} \langle R_{\{1\}}(n, n) | \kappa_0 R_{\{1\}}(n, n) \rangle \right) \end{aligned} \quad (4.13)$$

compensates (4.9) up to the following terms

$$\begin{aligned} & 2(a_{1,1} \langle \delta W(1, 1) | \kappa_0 R_{\{1\}}(1, 1) \rangle - a_{1,1} \langle \delta C(1, 1) | \kappa_0 R_{\{2\}}(1, 1) \rangle + \\ & + a_{1,1} \langle \delta W(1, 1) | \kappa_{++} R_{\{1\}}(0, 0) \rangle + a_{1,1} \langle \kappa_{++} \delta C(0, 0) | R_{\{2\}}(1, 1) \rangle - \\ & - \frac{a_{2,1}}{\alpha(2, 2)} \langle \delta C(2, 1) | \sigma_{(0)-}^2 \kappa_{++} R_{\{1\}}(1, 1) \rangle - \frac{a_{1,1}}{\alpha(1, 1)} \langle \delta C(1, 1) | \kappa_{--} \sigma_{(0)+}^2 R_{\{1\}}(2, 1) \rangle, \end{aligned}$$

which do not satisfy the extra field decoupling condition (4.1) and massless decomposition condition. Hence they should be compensated by S_{spec} of the form

$$S_{spec} = \frac{a_{1,1}}{\alpha(1, 1)} \int \epsilon_{l_1 \dots l_d} (\mu R_{\{1\}}^{l_1, l_2} R_{\{1\}} + \nu R_{\{1\}}^{l_1, b} R_{\{1\}}^{l_2, b}) e^{l_3} \dots e^{l_d}. \quad (4.14)$$

A somewhat involved calculation shows that the proper choice of μ and ν in (4.14) is

$$\nu = \frac{4}{(d-2)\alpha(1, 1)} \left(h(1, 1) \frac{d+2}{2(d-1)} - H(1, 0) - \lambda^2 \right), \quad \mu = \frac{4G_{(1)}(1, 0)g_{(1)}(0, 0)}{\alpha(0, 0)}. \quad (4.15)$$

It can be shown that (4.14) nevertheless can be represented in the regular form

$$S_{spec} = -\frac{a_{1,1}}{\alpha(1,1)} \langle R_{\{1\}}(1,1) | \kappa_0 R_{\{1\}}(1,1) \rangle - \frac{2a_{1,1}}{\alpha(1,1)} \langle R_{\{1\}}(1,1) | \kappa_{++}^{12} R_{\{1\}}(0,0) \rangle. \quad (4.16)$$

Let us note, that the conjugation relations (2.21) can be used to transform (4.13) to the form

$$\begin{aligned} S_{reg} &= - \sum_{reg} \langle \frac{a}{\alpha} R_{\{1\}} | \kappa R_{\{1\}} \rangle = \\ &= \sum_{n=2}^{s-1} \left(-\frac{a_{n,n}}{\alpha(n,n)} \langle R_{\{1\}}(n,n) | \kappa_0 R_{\{1\}}(n,n) \rangle - \frac{2a_{n,n}}{\alpha(n,n)} \langle R_{\{1\}}(n,n) | \kappa_{++}^{12} R_{\{1\}}(n-1,n-1) \rangle \right). \end{aligned} \quad (4.17)$$

S_{spec} (4.16) has the form of the term in brackets on the r. h. s. of (4.17) with $n = 1$.

The manifestly gauge invariant action for symmetric massive fields of any spin $s \geq 2$ is

$$\begin{aligned} S &= \sum_{1 \leq l \leq k \leq s-1} a_{k,l} \langle R_{\{2\}}(k,l) | R_{\{2\}}(k,l) \rangle - \\ &- \sum_{n=1}^{s-1} \left(\frac{a_{n,n}}{\alpha(n,n)} \langle R_{\{1\}}(n,n) | \kappa_0 R_{\{1\}}(n,n) \rangle + \right. \\ &\left. + \frac{2a_{n,n}}{\alpha(n,n)} \langle R_{\{1\}}(n,n) | \kappa_{++}^{12} R_{\{1\}}(n-1,n-1) \rangle \right). \end{aligned} \quad (4.18)$$

For the reader's convenience let us recall relevant notations. The curvatures R given in (3.19)-(3.22) are contracted by means of the scalar product $\langle \dots | \dots \rangle$ (2.18). The curvatures contain operators $\sigma = \sigma_-^1 + \sigma_+^1 + \sigma_-^2 + \sigma_+^2$ defined by (2.5)-(2.8) both for 0- and 1-forms and operators κ given in (3.26). The theory has the field redefinition freedom (2.16) independently for 0- and 1-forms. To fix it one should choose the scaling gauge. The type-I scaling gauge is (2.40). The type-II scaling gauge is (2.41), (2.42). Since, for the type-II scaling, $N_1(k,l)$ is k independent and $n_{(0)}(k,l)$ is l independent, we introduce notation $N_{(1)}(l) = N_{(1)}(k,l)$ and $n_{(0)}(k) = n_{(0)}(k,l)$. Note that $N_{(1)}(l) = -n_{(0)}(l-1)$. The coefficients in σ are determined by (2.29)-(2.33) where $H(k,l)$ and $h(k,l)$ are given by (2.45). The coefficients $a_{k,l}$ and α are given by (4.5), (3.24) for the type-I scaling gauge and by (4.7), (3.25) for the type-II scaling gauge.

The variation of (4.18) is

$$\begin{aligned} \delta S &= 2 \sum_{1 \leq k \leq s-1} a_{k,1} \left(\langle \sigma_{(1)+}^2 \delta W(k,0) | R_{\{2\}}(k,1) \rangle + \langle \delta W(k,1) | \sigma_{(1)+}^2 R_{\{2\}}(k,0) \rangle \right) + \\ &+ \frac{a_{1,1}\mu}{\alpha(1,1)} \int \epsilon_{l_1 \dots l_d} e^{l_3} \dots e^{l_d} \left(-\alpha(0,0) \delta C^{l_1, l_2} R_{\{2\}}(0,0) - \alpha(0,0) \delta W(0,0) R_{\{1\}}^{l_1, l_2} + \right. \\ &\left. + (\sigma_{(0)+}^2 \delta C)^{l_1, l_2} R_{\{1\}}(0,0) - \delta C(0,0) (\sigma_{(0)+}^2 R_{\{1\}})^{l_1, l_2} \right). \end{aligned} \quad (4.19)$$

The action (4.18) yields all equations used in the analysis of Section 3.3 of the dynamical content of the unfolded field equations. Namely, the second term of (4.19) yields the equations that express the first auxiliary fields in terms of the frame-like fields (3.40), (3.42), (3.47). The first term yields a non-trivial equations on the frame-like fields (3.43), (3.49). Other terms yield equations for the spin 0 and spin 1 fields analyzed in Section 3.3.

4.2 Spin 2 example

Let us consider the example of spin 2. Recall that terms involving 0-forms from the Weyl module (namely $C^{aa,bb}$ and $C^{aa,b}$) should be dropped from the action (4.18) which has the form

$$S = a_{1,1} \langle R_{\{2\}}(1,1) | R_{\{2\}}(1,1) \rangle - \frac{a_{1,1}}{\alpha(1,1)} \langle R_{\{1\}}(1,1) | \kappa_0 R_{\{1\}}(1,1) \rangle - \\ - 2 \frac{a_{1,1}}{\alpha(1,1)} \langle R_{\{1\}}(1,1) | \kappa_{++}^{12} R_{\{1\}}(0,0) \rangle. \quad (4.20)$$

Its variation is

$$\delta S = -2a_{1,1} \left(\langle \sigma_{(1)+}^2 \delta W(1,0) | R_{\{2\}}(1,1) \rangle - \langle \delta W(1,1) | \sigma_{(1)+}^2 R_{\{2\}}(1,0) \rangle \right) + \\ + \frac{a_{1,1}\mu}{\alpha(1,1)} \int \epsilon_{l_1 \dots l_d} e^{l_3} \dots e^{l_d} \left(-\alpha(0,0) \delta C^{l_1, l_2} R_{\{2\}}(0,0) - \alpha(0,0) \delta W(0,0) R_{\{1\}}^{l_1, l_2} + \right. \quad (4.21) \\ \left. + (\sigma_{(0)+}^2 \delta C)^{l_1, l_2} R_{\{1\}}(0,0) - \delta C(0,0) (\sigma_{(0)+}^2 R_{\{1\}})^{l_1, l_2} \right).$$

The first term gives the projection of (3.58) to $\square\square$. The second term gives (3.57). Other terms yield (3.53), (3.56) and traces of (3.54), (3.55). These are exactly the equations used in Subsection 3.4 to derive dynamical equations.

4.3 Massless and partially massless limits

To analyze the massless and partially limits it is convenient to use the type-II scaling gauge (2.41), (2.42). Let us fix the coefficient $a_{s-1,1}$ in (4.6) in such a way that the product $a_{k,1} \sigma_{(1)+}^2(k,0)$ in (4.19) be independent of m and λ . This insures non-zero variation in the flat massless case. Since $\sigma_{(1)+}^2(k,0)$ is proportional to $N_{(1)}^{-1}(0)$ (2.33), (2.41), (2.45) one should demand

$$a_{s-1,1} = a |N_{(1)}(0)|, \quad (4.22)$$

where a is a coefficient independent of m and λ .

For $\lambda^2 \neq 0$ and $m = 0$ the action still has the form (4.18) and its variation amounts to (4.19). As discussed in Section 3.5, the 1-form fields valued in $\mathbf{Y}(s-1, l)$ with arbitrary $l \leq s-1$ decouple from the other fields. The part of the action for these fields has the form

$$S_{m=0} = a_{s-1,1} \sum_{1 \leq l \leq s-1} \langle R_{\{2\}}(s-1, l) | R_{\{2\}}(s-1, l) \rangle \quad (4.23)$$

Being formulated in terms of the 1-forms $W^{a(s-1), b(l)}$ this is the action of [19] for a spin s massless particle in $(A)dS_d$. The leftover part of (4.18) is the action of a massive spin $(s-1)$ particle.

As discussed in Section 3.5, at $\lambda^2 = 0$, 1-forms decompose into s sets, each constituted by the fields valued in $\mathbf{Y}(s'-1, l)$ with $l \leq s'-1$ and various fixed $s' < s$. The action (4.18) also decomposes into s parts. The condition (4.22) guarantees that the variation is different from zero. Although the coefficients $a_{k,l}$ that have the form (2.41), (2.42), (4.7)

$$a_{k,l} = \frac{z(k, l)}{\left| \prod_{i=0}^{l-1} (m^2 + \lambda^2(s-l-1)(s+l+d-4)) \right|} \quad (4.24)$$

are singular in the flat massless limit $\lambda \rightarrow 0$, $m \rightarrow 0$, the related terms in the action (4.18) form a total derivative as is obvious from the fact that they do not contribute to the variation of the action. Dropping these terms, the action remains gauge invariant up to total derivative terms. However, it cannot be written in terms of manifestly gauge invariant curvatures for $m = \lambda = 0$, giving a combination of the actions found originally in [17].

In the partially massless limit, the action (4.18) also decomposes into two parts in agreement with the pattern described in Section 3.5. It yields equations for a partially massless field of spin s and depth t and for a massive spin t field of mass m_s (2.46).

In the type-II scaling gauge the partially massless part of the action contains the coefficients $a_{k,l}$ for $l \geq t+1$, that tend to infinity at $m \rightarrow m_t$ although the variation (4.19) remains finite. Analogously to the flat massless case, the singular terms combine into total derivatives and can be dropped from the action. The remaining part of the action is non-singular and yields correct equations. However, it cannot be written in terms of the curvatures. Note that this does not contradict to the results of [21], where the action for a single partially massless field is given in terms of curvatures because our description involves non-minimal curvatures that differ from those of [21]. (The main difference is that $\sigma_{(1)-}^2(k, t+1)$ (3.65) is nonzero for the non-minimal curvatures but vanishes in the curvatures of [21].)

5 Conclusion

In this paper we found the explicit form of (Stueckelberg) gauge invariant linearized curvatures for symmetric massive HS fields in d -dimensional Minkowski and $(A)dS$ space. In terms of these curvatures we constructed the manifestly gauge invariant action as well as full unfolded field equations for free symmetric massive HS fields.

An interesting problem for the future is to extend obtained results to general massive mixed symmetry fields. Note that the unfolded equations for mixed-symmetry fields have been recently analyzed in [29] via radial reduction of the frame-like formulation of mixed symmetry fields in Minkowski space developed by Skvortsov [34]. The manifestly gauge invariant action for mixed-symmetry massive fields in $(A)dS$ remains to be worked out.

The challenging problem is to extend obtained results to interacting massive HS fields that may help to establish the correspondence between HS gauge theory and String Theory.

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Appendix A: Conventions

In this paper we use the following conventions. Space-time is d -dimensional Minkowski or $(A)dS$ space. Both base indices μ, ν, \dots (*i.e.*, indices of differential forms) and fiber indices a, b, \dots run from 0 to $d - 1$. The fiber space is endowed with the mostly minus Minkowski metric η_{ab} . Upper and lower indices denoted by the same letter are contracted. Upper or lower symmetrized indices can be also denoted by the same letter. A number of symmetrized indices is often indicated in brackets by writing e.g. $a(k)$ instead of $(a_1 \dots a_k)$. In this paper we deal with traceless tensors that possess symmetries of two-row Young tableaux. Indices of the first and second rows are separated by comma, *i.e.*, the notation $C^{a(k),b(l)}$ automatically implies that

$$C_n^{na(k-2),b(l)} = 0, \quad C_n^{a(k-1),nb(l-1)} = 0, \quad C^{a(k),n}{}^{nb(l-2)} = 0, \quad C^{a(k),ab(l-1)} = 0.$$

Sometimes instead of writing indices of tensors that possess symmetries of a two-row Young tableau, we use the shorthand notation indicating the lengths of rows in brackets. For instance

$$C(k, l) \sim C^{a(k),b(l)}.$$

We often deal with similar objects for 0- and 1-forms. To distinguish between these objects we endow them with subscripts (0) and (1) which refer to the degree of a differential form they act on. No subscript means that the formula is the same for 0- and 1-forms.

Appendix B: Arbitrary scaling

Here we present some useful expressions valid for arbitrary $N_{(0)}(k, l)$, $n_{(0)}(k, l)$, $N_{(1)}(k, l)$ and $n_{(1)}(k, l)$.

The coefficients $f_{(0)}(s, l)$ in $\sigma_{(0)-}^1(s, l)$ (3.17), which are responsible for gluing of the Weyl module to the sector of mixed 0- and 1-forms, are defined by (2.15) up to an overall factor

$$\frac{f_{(0)}(s, l)}{f_{(0)}(s, l-1)} = \frac{F_{(0)}(s, l)}{F_{(0)}(s-1, l)} \frac{s-l+1}{s-l}, \quad l < s \quad (\text{B.1})$$

and $f_{(0)}(s, s) = 0$ (2.10).

The compatibility condition (3.5) with $p = 0$ implies

$$\alpha(k, l+1) = \alpha(k, l) \sqrt{\frac{N_{(1)}(k, l)}{N_{(0)}(k, l)}}, \quad \alpha(k+1, l) = \alpha(k, l) \sqrt{\frac{n_{(1)}(k, l)}{n_{(0)}(k, l)}}. \quad (\text{B.2})$$

Eq. (B.2) expresses any $\alpha(k, l)$ in terms of $\alpha(0, 0)$

$$\alpha(k, l) = \alpha(0, 0) \sqrt{\frac{\prod_{i=0}^{k-1} n_{(1)}(i, 0) \prod_{j=0}^{l-1} N_{(1)}(k, j)}{\prod_{i=0}^{k-1} n_{(0)}(i, 0) \prod_{j=0}^{l-1} N_{(0)}(k, j)}}. \quad (\text{B.3})$$

(Note that Eqs. (B.2) are consistent by virtue of (2.25).) Eq. (4.4) can be used to express $a_{k,l}$ in terms of $a_{s-1,1}$:

$$a_{k,l} = a_{s-1,1} \frac{\prod_{i=1}^{l-1} N_{(1)}(s-1, i)}{\prod_{j=k}^{s-2} n_{(1)}(j, l)}. \quad (\text{B.4})$$

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